Interdisciplinary Lively Applications Project

Path Integrals in Physics

Work and Electrostatic Potential: Path Integrals in Introductory Physics

An Interdisciplinary Lively Application Project (ILAP)

Authors: G. Raymond Brown, Associate Professor of Physics
Alexander Fluellen, Professor of Mathematical Sciences

Department of Physics and Department of Mathematical Sciences, Clark Atlanta University, Atlanta, Georgia

Editor: David C. Arney, Head, Department of Mathematics, United States Military Academy

Mathematics Classifications: Calculus

Prerequisite Skills:

1. Vector Analysis: addition, inner products, outer products, some differential operators
2. Integral Calculus: antiderivatives of simple polynomials
3. Drawing graphs of functions
4. Some familiarity with basic physics concepts: forces, the electrostatic force given by Coulomb's Law, work and energy

Physical Concepts Examined:

1. The physical definition of the work done by a force
2. The distinction between conservative and non-conservative forces and fields.
3. Association of potential energy with conservative forces
4. Definition of the electrostatic potential as a potential energy per unit charge
5. Methods of calculating the electrostatic potential

The object of this exercise is to lead the student to a connection between some rather abstract constructs in physics and mathematics, that is, work and the electrostatic potential, and the geometry of three-dimensional Euclidean space. An understanding by the student of the underlying geometry of the concepts is a major goal in teaching the material, which appears in introductory calculus-based physics courses in mechanics and electromagnetism. The traditional teaching tools of textbooks, lectures including notes and drawings on a blackboard, and overhead projections of drawings, provide hints of the three-dimensional geometry, but require that the student have or develop the ability to visualize that geometry.

In response to the "visualization difficulty", we developed the idea of a new tool consisting of convincingly three-dimensional animated cartoons depicting the geometrical content of the material, thereby providing the student with the visualization itself. A first step in this direction is the storyboard presentation incorporated in this project. The storyboard is provided as background material to help the student solve the problems in the activities given at the beginning of the project. It is also a valuable learning tool for all students.

Each page of the storyboard is divided into a left and right vertical column. In the left column are still cartoon panels consisting primarily of drawings illustrating the concepts. The right column consists of narrative text and equations that develop the concepts. The storyboard is to be read almost like a newspaper comic strip, with the narrative in the right column forming an extended caption for the cartoon panels.

In this ILAP, we address a set of activities that are "everyday" physics exercises for a student in the introductory courses. That is, they are basic exercises that can be solved in a few minutes or hours. They might, indeed, appear in sets of homework problems assigned in the course.

We emphasize basic activities because they most clearly ask the student to display his or her answers to the two most important questions in the technical practice of a prospective professional scientist or engineer:

1. Do I understand what the theoretical equations *mean*, when used as mathematical models of the behavior of physical reality?

2. Have I learned to *use* the theoretical equations to address and solve physical problems?

When the answer to both of these questions is a resounding "Yes!", then the student is prepared to become a professional. The difficult and complex problems that will inevitably arise in a professional career can be addressed with confidence.

On the other hand, the exercises are chosen and posed to emphasize the use and properties of path integrals, a type of mathematical object that plays an important role in our understanding of physics and engineering beyond the introductory level. Most of the exercises address conservative forces, such as the gravitational force and the electrostatic force, for which path integrals are particularly simple. The term "conservative force" is a handy piece of technical jargon, most simply defined as a force whose curl is zero, that is, one that has no rotational component. Most (all?) fundamental physical forces have this property. It is more carefully defined in the storyboard, on page 15.

Given the prerequisite skills in vector analysis and calculus, the student can find in the storyboard all the concepts needed to successfully address the activities. The student should carefully read the storyboard at the beginning of the project, then refer to it for help with any difficulties found in addressing the activities.

The activities represent foundations for more advanced study in mechanics, orbital mechanics, electrical engineering, physics, and chemistry. For example, Activity 7 might be looked upon by a chemistry student as a crude model of a polar molecule, for which the calculated potential and electric field mediate chemical interaction with other molecules. An enthusiastic student might undertake the exercise of elaborating the dipole by separating the positive charge into two equal particles, displaced from the vertical axis by equal angles, thereby generating a (still crude and thoroughly classical) model of a water molecule, and repeating the calculations for the new configuration. Many of the properties and symmetries of the results for the dipole survive such an elaboration.

Activities

1. The Local Gravitational Force

The local gravitational force near the surface of the earth is given by $\boldsymbol{F}_g = mg\left(-\hat{\boldsymbol{k}}\right)$, where m is the mass of a particle on which the force acts, $g = 9.80\,\frac{m}{s^2}$ is the local gravitational acceleration, and $\hat{\boldsymbol{k}}$ is the vertical unit vector. Explicitly show that this force is conservative (see page 15 for the definition of a conservative force) by performing each of the following three calculations.

(a) Show that the work done by this force on a particle of mass m in going from the point $\boldsymbol{A} = \left(x_A, y_A, z_A\right)$ to the point $\boldsymbol{B} = \left(x_B, y_B, z_B\right)$ is independent of the path chosen between the points. [Hint: Because $\boldsymbol{F}_g$ is constant over the path and the dot product operation is distributive over a sum, the form for the work reduces to $\mathcal{W} = \boldsymbol{F}_g \bullet \int_A^B d\boldsymbol{s}$. Consider the meaning of the remaining path integral. This easily generalizes to a proof that any constant force is conservative.]

(b) Show that the work done by this force on a particle of mass m is zero for one trip around the vertical circle described by $x^2 + z^2 = R^2$, where R is the constant radius of the circle.

(c) Show that $\nabla \times \boldsymbol{F}_g = 0$. [Hint: What are the partial derivatives of a constant?]

2. The Universal Gravitational Force

The universal gravitational force is given by $\boldsymbol{F}_G = G\dfrac{m_1 m_2}{r^2}\left(-\hat{\boldsymbol{r}}\right)$, where

$G = 6.673 \times 10^{-11}\,\frac{m^3}{s^2 \cdot kg}$ is the universal gravitational constant, m_1 and m_2 are the masses of the attracting bodies, r is the distance between the centers of mass of the bodies, and $\hat{\boldsymbol{r}}$ is the radial unit vector from the center of one of the bodies. (Note that it doesn't matter which mass you call m_1 or m_2. For either choice, the force on the chosen body involves the unit vector away from the *other* body.) Explicitly show that this force is conservative by performing each of the following calculations.

(a) Show that the work done by the gravitational force on a satellite of mass m in a circular orbit around the earth, mass $m_E >> m$, is zero.

(b) Show that $\nabla \times \boldsymbol{F}_G = 0$. (In spherical coordinates, the curl of a vector $\boldsymbol{v}$ is given by

$$\nabla \times \boldsymbol{v} = \frac{1}{r\,sin\theta}\left[\frac{\partial}{\partial\theta}\left(v_\phi\, sin\theta\right) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\boldsymbol{e}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{o}}\text{ .)}$$

3. The Electrostatic Field of a Single Point Charge

The form for the electrostatic field of a point charge q is given by Coulomb's Law, $\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$, where ε_0 is the permittivity of free space, $\varepsilon_0 = 8.854 \times 10^{-12}\ \frac{C^2}{N \cdot m^2}$, and $\hat{\boldsymbol{r}}$ is the radial unit vector from the point charge. Explicitly show that this is a conservative field by the following calculations.

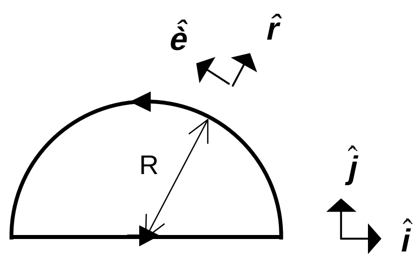
(a) Show that $\oint \boldsymbol{E} \bullet d\boldsymbol{s} = 0$ over any closed path in three-dimensional space. Use

$$d\boldsymbol{s} = dr\hat{\boldsymbol{r}} + r\, sin\,\theta d\phi\hat{\boldsymbol{\sigma}} + r d\theta\hat{\boldsymbol{e}}.$$

(b) Show that $\nabla \times \boldsymbol{E} = 0$.

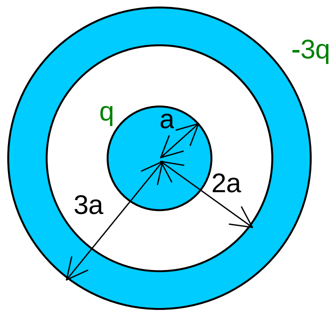
4. The Contact Frictional Force

The force of friction between a sliding object of mass m and a horizontal surface on which the object slides is modeled by $\boldsymbol{f}_k = \mu_k mg(-\hat{\boldsymbol{v}})$, where g is the local gravitational constant, μ_k is the coefficient of kinetic friction, and $(-\hat{\boldsymbol{v}})$ is a unit vector opposite to the velocity vector $\boldsymbol{v}$. Calculate the work done by this force around the closed path shown in the drawing and show that this frictional force is non-conservative. On the straight-line part of the closed path, use $d\boldsymbol{s} = dx\hat{\boldsymbol{i}}$, and on the semicircular return path, use $d\boldsymbol{s} = R d\theta\hat{\boldsymbol{e}}$. Then independently calculate $\nabla \times \boldsymbol{f}_k \neq 0$. [Hint: Note that the only part of $\boldsymbol{f}_k$ that depends on the spatial coordinates is its unit vector.]



5. The Potential of a Spherically Symmetric Pair of Conductors

Consider a charge distribution consisting of a conducting sphere of radius a carrying a charge q, and a concentric conducting shell, inner radius $2a$ and outer radius $3a$, carrying a charge of $-3q$.



Take the reference point of the electrostatic potential such that $V(r=\infty)=0$, where r is the radial distance variable from the center of the charge distribution. The electric field due to this charge distribution is

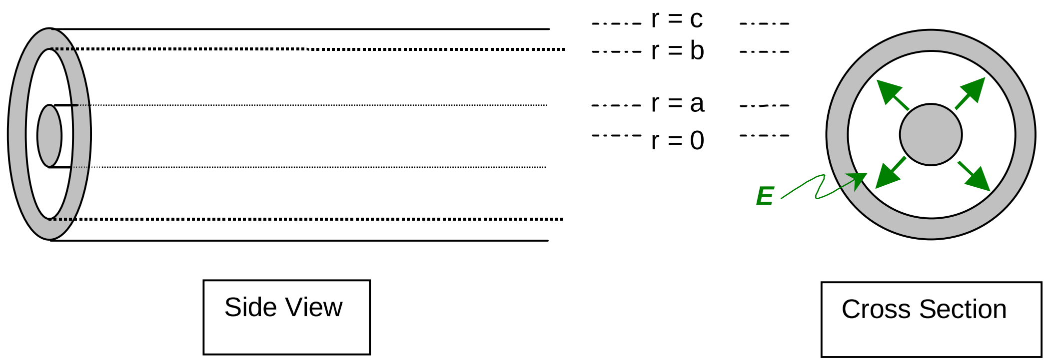
$$\boldsymbol{E}(r)=\begin{cases} 0\,, & r<a. \\ \dfrac{q\hat{\boldsymbol{r}}}{4\pi\varepsilon_0 r^2}\,, & a\le r\le 2a. \\ 0\,, & 2a<r<3a. \\ \dfrac{q(-\hat{\boldsymbol{r}})}{2\pi\varepsilon_0 r^2}\,, & r\ge 3a. \end{cases}$$

(a) Find the potential $V(r)$ for all values of r in the range $[0,\infty)$. Specify the values $V(3a), V(2a), V(a),$ and $V(0)$.

(b) Use your results to plot the variation of both $E(r)$ and $V(r)$ on the same graph, showing specific values at the conductor surfaces.

6. The Potential inside a Coaxial Cable

Consider a long, straight coaxial cable. The cable consists of a central conducting rod of radius a, with a linear charge density λ, and an outer conducting cylindrical shell, with inner radius b and outer radius c, with a linear charge density $-\lambda$.



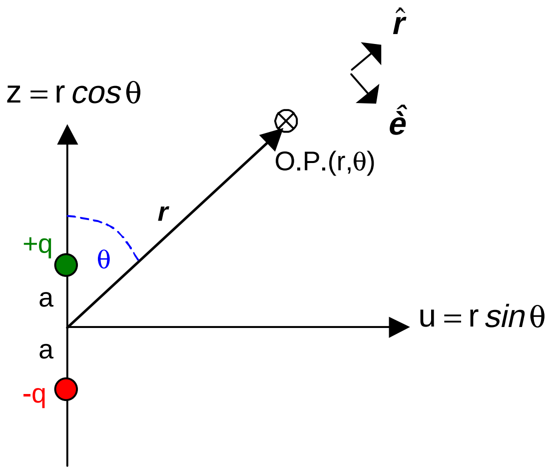
All of the electric charge lies on the surfaces of the conductors, and the electric field within the cable is given by $\boldsymbol{E}(r)=\begin{cases}\dfrac{\lambda}{2\pi\varepsilon_0 r}\hat{\boldsymbol{r}}, & a \le r \le b \\ 0, & r < a,\ r > b\end{cases}$. Take the reference point for the potential to be the outer surface of the cable, where $V(c)=0$.

(a) Find the potential $V(r)$ for all values of r in the range $[0,c]$. Specify the values $V(b)$, $V(a)$, and $V(0)$.

(b) Use your results to plot the variation of both $E(r)$ and $V(r)$ on the same graph, showing specific values at the conductor surfaces.

7. The Potential and Electric Field of a Dipole

Consider a dipole consisting of equal and opposite charges q and $-q$, separated by a distance $2a$. The electric field due to this charge configuration has cylindrical symmetry about the line connecting the charges. Therefore we choose an origin at the midpoint of the line connecting the charges, and designate the location of our observation point (marked as O.P. in the figure below) by the coordinates r and θ, as shown in the figure.



(a) Use the form of the potential determined by superposition of multiple point charges, $V = K\sum_{n=1}^{N}\frac{q_n}{r_n}$, where K is Coulomb's constant $K = (4\pi\varepsilon_0)^{-1}$ and N is the number of point charges in the charge distribution. Show that, at the observation point,
$V(r,\theta) = Kq\left[\left(r^2 + a^2 - 2ar\,cos\theta\right)^{-1/2} - \left(r^2 + a^2 + 2ar\,cos\theta\right)^{-1/2}\right]$.

(b) By explicitly taking the limit $\lim_{r\to\infty}[V(r,\theta)]$, show that $V(r = \infty) = 0$, in agreement with the derivation of the form $V = K\sum_{n=1}^{N}\frac{q_n}{r_n}$.

(c) Show that the midplane at $z = 0$ $\left(\theta = \frac{\pi}{2}\right)$ is a plane of odd mirror symmetry (antisymmetry) for the potential, that is, that $V(r,\pi - \theta) = -V(r,\theta)$.

(d) Show that the electric field for the dipole is given by

$$\boldsymbol{E}(r,\theta) = Kq\left\{\begin{array}{l}\left[\frac{(r - a\,cos\theta)}{\left(r^2 + a^2 - 2ar\,cos\theta\right)^{3/2}} - \frac{(r + a\,cos\theta)}{\left(r^2 + a^2 + 2ar\,cos\theta\right)^{3/2}}\right]\hat{\boldsymbol{r}} \\ + (a\,sin\theta)\left[\left(r^2 + a^2 - 2ar\,cos\theta\right)^{-3/2} + \left(r^2 + a^2 + 2ar\,cos\theta\right)^{-3/2}\right]\hat{\boldsymbol{e}}\end{array}\right\}$$

, by using the gradient operator in spherical coordinates,

$$\boldsymbol{E}(r,\theta) = -\nabla V(r,\theta) = -\left[\frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{e}} + \frac{1}{r\,sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{o}}\right].$$

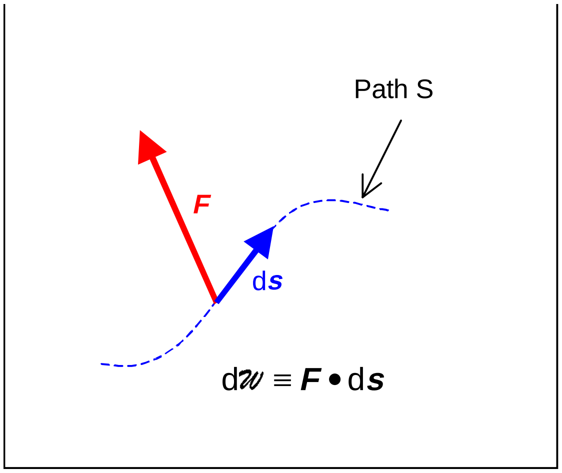
(e) Show that the electric field has the same odd mirror symmetry about the midplane as does the potential. That is, as $z \to -z$ $(\theta \to \pi - \theta)$, E_z (or E_θ) remains unchanged, while E_u (or E_r) changes sign. Note that $V\left(r,\frac{\pi}{2}\right) = 0$, so the potential everywhere in the midplane is zero, while $\boldsymbol{E}\left(r,\frac{\pi}{2}\right)$ is never zero. How do you interpret this observation? Locate and determine the maximum value that $\boldsymbol{E}\left(r,\frac{\pi}{2}\right)$ has in the midplane.

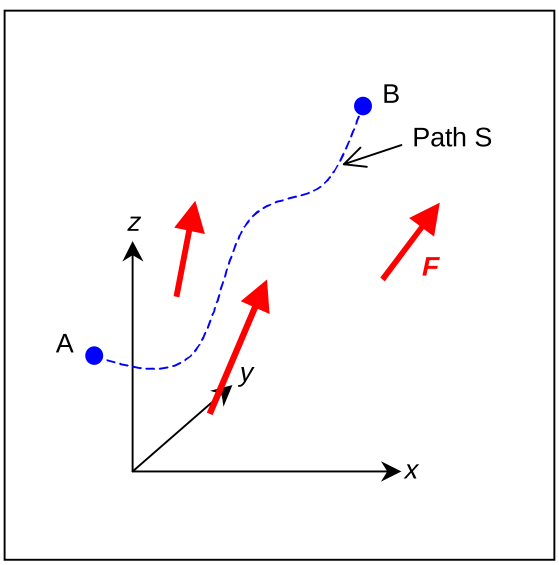
(f) Explicitly show by integration of $\boldsymbol{E}(\mathrm{r},\theta)$ along a path at any constant value of θ from $\mathrm{r}=\infty$ to an arbitrary finite value of r that $V(\mathrm{r},\theta)=-\int_{\infty}^{\mathrm{r}}\boldsymbol{E}\bullet d\boldsymbol{s}$ yields the same form as was found in part (a).

(g) Finally, obtain approximate forms of the potential and the electric field for distances large compared to the separation of the charges, that is, for $\mathrm{r} >> \mathrm{a}$, by expanding the radicals with the help of the binomial approximation and discarding all but the leading terms. (By the binomial approximation, for $\delta << 1$, $(1+\delta)^n \approx 1+n\delta$.) We add a tilde to represent the approximate quantities at large distance. Show that $\tilde{V}=\frac{2Kqa\,cos\theta}{r^2}$, and $\tilde{\boldsymbol{E}}=\frac{2Kqa}{r^3}\left(3\,cos\theta\,\hat{\boldsymbol{r}}+sin\theta\,\hat{\boldsymbol{\theta}}\right)$. Note and comment on the fact that these approximate expressions have lost an important property of the exact expressions, specifically that $\tilde{\boldsymbol{E}} \neq -\nabla\tilde{V}$.

The definitions of Work and the Electrostatic Potential: Path Integrals

Cartoon Boards





The definitions of Work and the Electrostatic Potential: Path Integrals

Narrative

Consider the action of a force $\boldsymbol{F}$ on some particle moving along an element of a path $d\boldsymbol{s}$. Because the directions of the force and the path element may differ, we use the projection of the force in the direction of the path element. The element of work done by the force, $d\mathcal{W}$, is defined as the dot product of the two vectors:

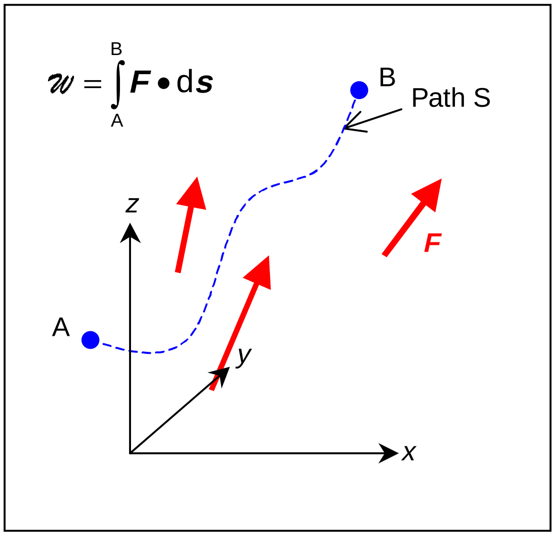
$$d\mathcal{W} \equiv \boldsymbol{F} \bullet d\boldsymbol{s}.$$

Note that work has units of energy.

$$\{\mathcal{W}\} = \{F \cdot \text{length}\} = \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2} = \text{Joules}.$$

Now suppose that a particle is moved from some initial location, which we will call point A, to a final location, point B, along a path S, while under the influence of some force $\boldsymbol{F}$. To find the total work done by the force during this process, we must sum all the elements of work, giving us the integral:

$$\mathcal{W} = \int_A^B \boldsymbol{F} \bullet d\boldsymbol{s}.$$



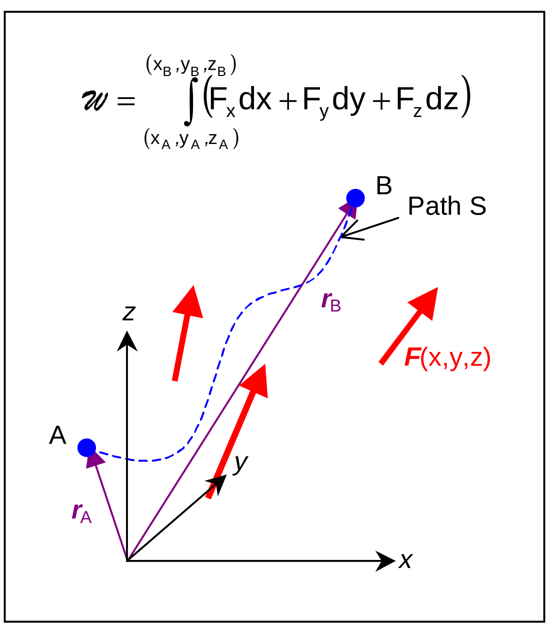
So determination of the total work requires a calculation of the value of this integral. Let's begin by expressing the vectors in Cartesian components, and expanding the scalar product:

$$\boldsymbol{F} = F_x \hat{\boldsymbol{i}} + F_y \hat{\boldsymbol{j}} + F_z \hat{\boldsymbol{k}},$$
$$d\boldsymbol{s} = dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}},$$
$$\boldsymbol{F} \bullet d\boldsymbol{s} = F_x dx + F_y dy + F_z dz.$$

Note that the initial and final locations are vectors:

$$\boldsymbol{r}_A = x_A \hat{\boldsymbol{i}} + y_A \hat{\boldsymbol{j}} + z_A \hat{\boldsymbol{k}},$$
$$\boldsymbol{r}_B = x_B \hat{\boldsymbol{i}} + y_B \hat{\boldsymbol{j}} + z_B \hat{\boldsymbol{k}},$$

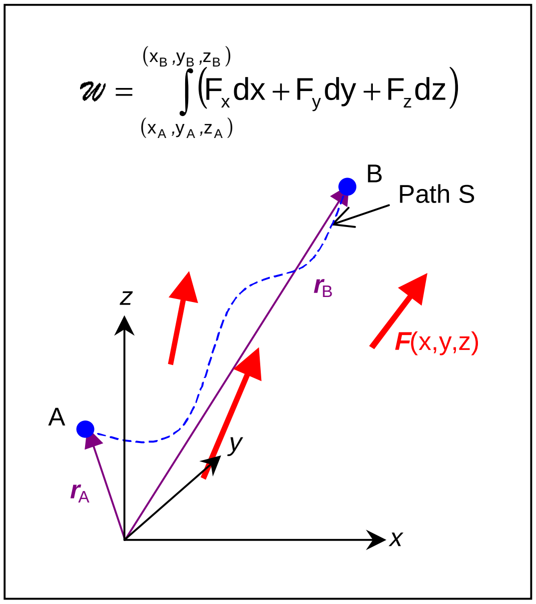
where we have chosen an (arbitrarily placed) origin and Cartesian coordinates.



We can write the work done by the force as

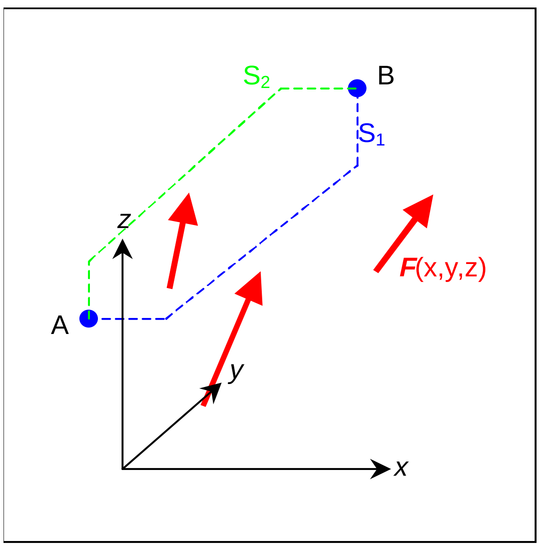
$$\mathcal{W} = \int_{(x_A,y_A,z_A)}^{(x_B,y_B,z_B)} \left[F_x(x,y,z)dx + F_y(x,y,z)dy + F_z(x,y,z)dz\right].$$

Here we have explicitly shown that each of the force components might in general be a function of all three coordinates. (The force could also depend on time, but this would make no difference in our calculation of work, which is defined only in terms of spatial coordinates. Any time dependence of the force would simply appear as a time dependence of the work.)



This looks like quite a mess to calculate, because each force component can depend on all three coordinates, and the equation of the path may be a complex function of the coordinates. Let's try to make it simpler. We will find that the physical forces of gravity and electrostatics "make work simple"!

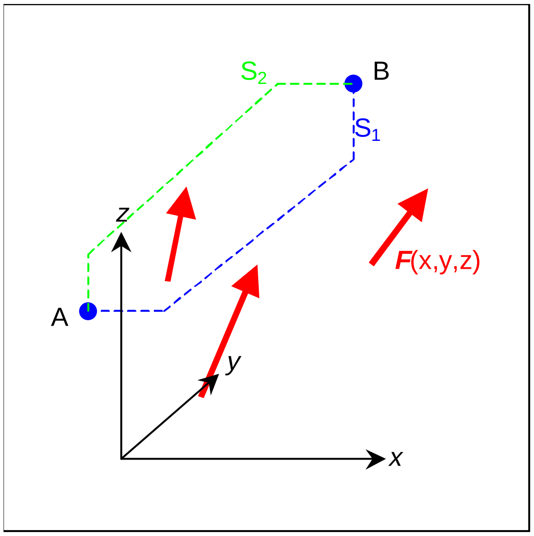
Note that the calculation is much less complex if one can choose a simple path on which to do the calculation.



[As the narration and text proceed, a blue or green dot moves along the appropriate path in the cartoon, and the corresponding integral appears overlaying the cartoon.]

For example, the path S_1 in blue consists of 3 straight line segments: first a line on which the *y*- and *z*- coordinates are held constant at their values for point A while *x* varies from x_A to x_B, then *x* and *z* are held constant while *y* varies from y_A to y_B, and finally *x* and *z* are held constant at their values for point B while *z* varies from z_A to z_B. The path S_2 in green is another such possible simple path. For the work along S_1:

$$\mathcal{W}_1 = \int_{x_A}^{x_B} F_x(x, y_A, z_A)dx + \int_{y_A}^{y_B} F_y(x_B, y, z_A)dy + \int_{z_A}^{z_B} F_z(x_B, y_B, z)dz.$$



The calculation has been reduced to a sum of 3 ordinary one-dimensional integrals, because our choice of simple line segment paths aligned with the coordinate axes reduced the other coordinate variables in each integral to constant values.

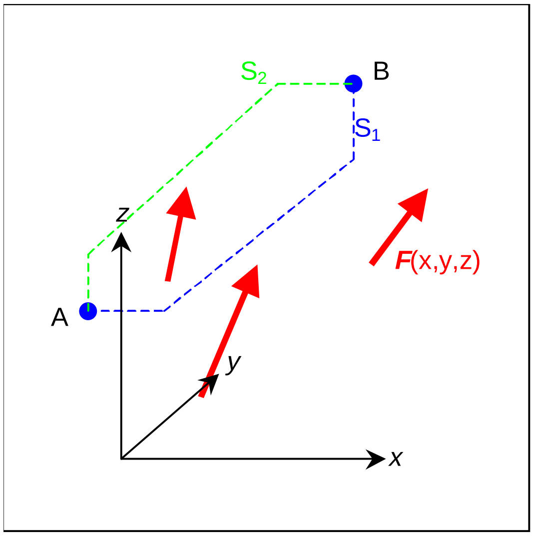
We could do the same thing for path S_2:

$$\mathcal{W}_2 = \int_{x_A}^{x_B} F_x(x, y_B, z_B)dx + \int_{y_A}^{y_B} F_y(x_A, y, z_B)dy + \int_{z_A}^{z_B} F_z(x_A, y_A, z)dz.$$

In general, motion along the two different paths would yield different values of the work: $\mathcal{W}_2 \neq \mathcal{W}_1$. This is because, in general, $F_x(x, y_A, z_A) \neq F_x(x, y_B, z_B)$, and also for the other integrands.

Simplifying the path made the calculation much easier, by forcing variables other than the integration variable in each of the three integrals to specific constant values.

Putting a twist on this, let's introduce a constraint on the force, instead of one on the path. Seeking simplicity, we want to avoid variables that make the calculation complex. Let's explore what happens if we simply "outlaw" those variables.



[As before, dots travel along the paths as the text and narration describe the integration.]

Let the force be such that each component depends only on the corresponding coordinate, so that

$$F_x = F_x(x),\ F_y = F_y(y),\ F_z = F_z(z).\ [1]$$

Now if we calculate the work between points A and B, using the same two paths, we find:

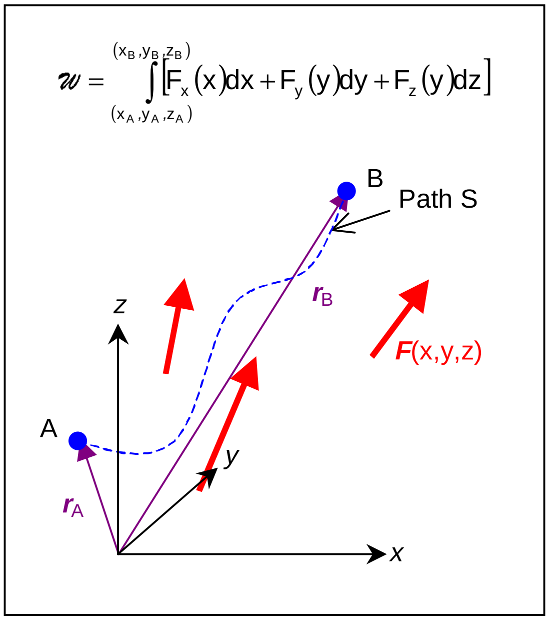
$$\mathcal{W}_1 = \int_{x_A}^{x_B} F_x(x)dx + \int_{y_A}^{y_B} F_y(y)dy + \int_{z_A}^{z_B} F_z(z)dz, \text{ and}$$

$$\mathcal{W}_2 = \int_{x_A}^{x_B} F_x(x)dx + \int_{y_A}^{y_B} F_y(y)dy + \int_{z_A}^{z_B} F_z(z)dz. \text{ Thus}$$

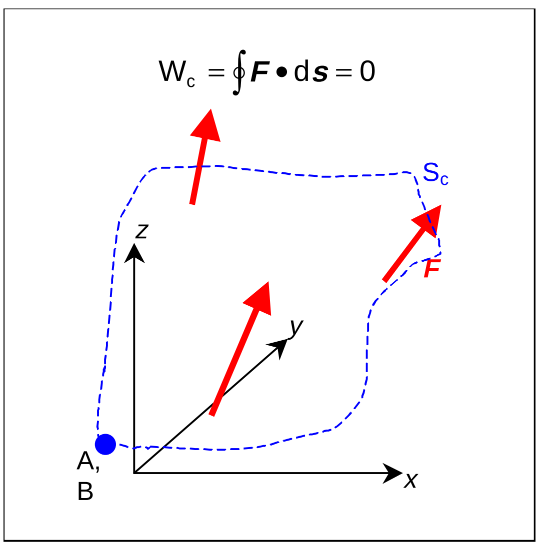
$$\mathcal{W}_1 = \mathcal{W}_2 .$$

For our constrained force, the form for the work immediately reduces to the sum of three independent one-dimensional integrals. Furthermore, the calculated work is the same for both paths, and depends only on the location of the starting and ending points. The calculation simply doesn't depend on the path, and we get the same value of work for *any path whatsoever* from point A to point B.

A force obeying this restriction ([1]) has the remarkable property that the work done by the force between any two points depends only on the location of the two points, and not on the path taken between them.



We need to keep in mind that we have restricted the form that the force can take to a very special case. Still, let's explore further the consequences of a force that leads to a value for the work that is independent of the path taken from an initial location to a final location.

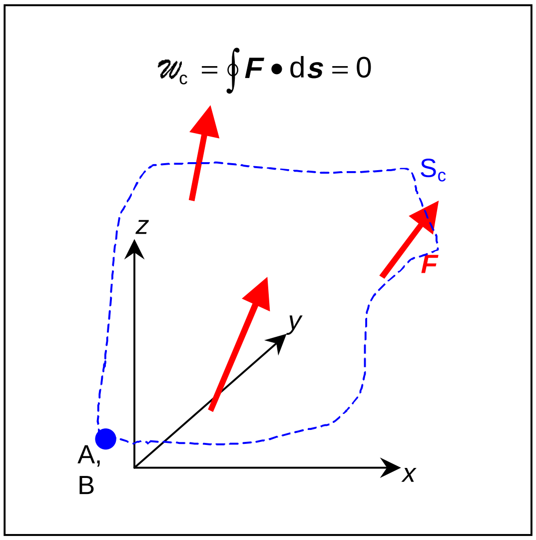


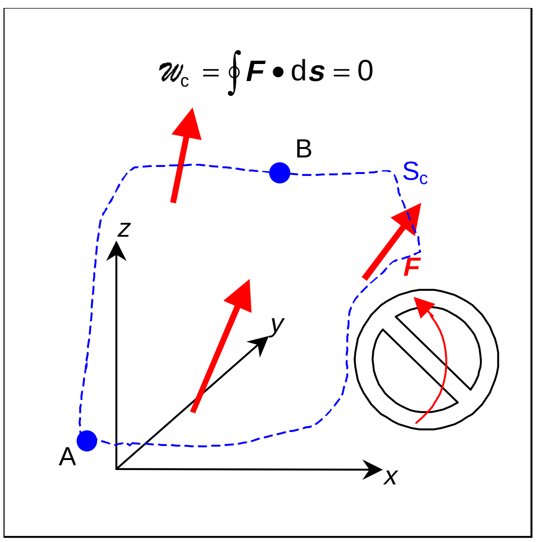
[The blue dot moves entirely around the closed path during the narration.]

There is an important corollary to our finding the value of the path integral to be independent of the path. *If the value of the work done is independent of the path, then the work depends only on the difference between the coordinates of the initial and final points.* If we make the initial and final points identical, then the work done is necessarily zero. If the work is calculated over any *closed path*, that is, one for which the initial and final points are the same, then the work done is identically zero. Using a subscript C to represent the work around a closed path, we have that:

$$\mathscr{W}_c = \oint \boldsymbol{F} \bullet d\boldsymbol{s} = 0 .$$

We can also show the converse that, if the work done around a closed path is zero, then the work done between any two points on the path depends only on the locations of the points.



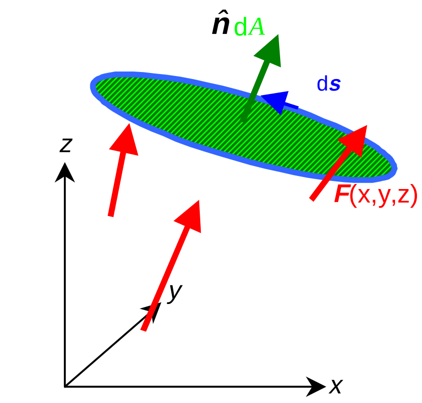


Consider an arbitrary intermediate point B on a closed path. The work done in going from A to B along any arbitrary path is some value $\mathcal{W}_{AB}$. Returning to point A by another arbitrary path completes a closed path for which $\mathcal{W}_c = 0$. Thus $\mathcal{W}_{AB} = -\mathcal{W}_{BA}$ for whatever intermediate paths are taken, so $\mathcal{W}_{AB}$ is independent of the path

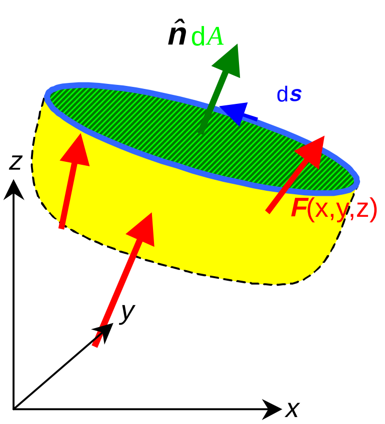
The condition imposed on the force to get the simplification, that each component of the force be independent of the other coordinates in three-dimensional space, is *more stringent than necessary* to get this very pretty result. The necessary and sufficient condition on the force so that the work done be independent of path is that the force have *no rotational part*. The closed path expression $\mathcal{W}_c = \oint \boldsymbol{F} \bullet d\boldsymbol{s} = 0$ hints in this direction.

Mathematically, the force has no rotational part if:

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0.$$



$$\mathcal{W}_c = \oint \boldsymbol{F}\bullet d\boldsymbol{s} = \int_{\substack{\text{bounded}\\ \text{surface}}} (\nabla\times\boldsymbol{F})\bullet\hat{\boldsymbol{n}}\,dA = 0$$



The same circular path (in blue) bounds both the plane circular surface (in green stripes) and the bowl-shaped surface (in yellow).

Or more compactly: $\nabla\times\boldsymbol{F}=0$. This is read as "The **curl** of $\boldsymbol{F}$ is zero."

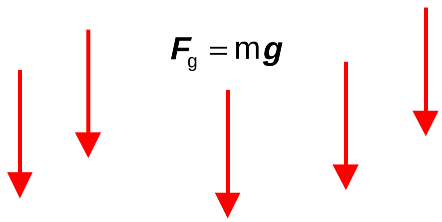
($\nabla \equiv \hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}$. For a review of the vector calculus, see a calculus textbook, for example Finney and Thomas {1}, Chapter 12 and page 985 and following. An excellent discussion from the physicist's point of view is found in Griffiths {2}, Chapter 1.)

The relationship between the closed path integral expression for work and the curl of the force is a specific case of Stokes' Theorem: $\oint \boldsymbol{F}\bullet d\boldsymbol{s} = \int_{\substack{\text{bounded}\\ \text{surface}}} (\nabla\times\boldsymbol{F})\bullet\hat{\boldsymbol{n}}\,dA$.

Note that for any single closed loop, there are an infinite number of open surfaces that are bounded by the loop. (Two are shown.)

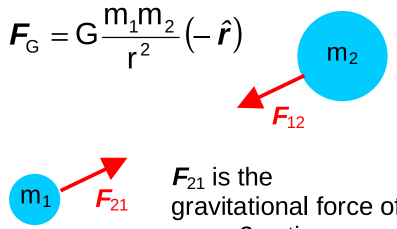
If $\oint \boldsymbol{F}\bullet d\boldsymbol{s} = 0$ for **every** closed loop then $\nabla\times\boldsymbol{F}=0$ is required to make the surface integral always zero.

To give a name to the class of forces for which work is independent of path, forces that have this property are called **conservative forces.**



A mass anywhere in the near vicinity of the earth's surface is subject to a downward gravitational force.

$$\boldsymbol{F}_G = G\frac{m_1 m_2}{r^2}(-\hat{\boldsymbol{r}})$$



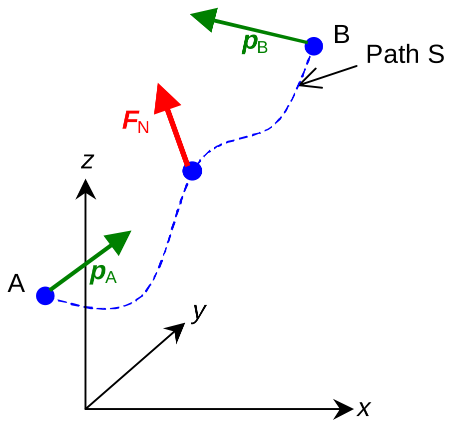
$\boldsymbol{F}_{21}$ is the gravitational force of mass 2 acting on mass 1, and vice versa.

The local gravitational force, $\boldsymbol{F}_g = m\boldsymbol{g}$, the universal gravitational force, $\boldsymbol{F}_G = G\frac{m_1 m_2}{r^2}(-\hat{\boldsymbol{r}})$, and the electrostatic force $\boldsymbol{F}_E = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$ are all examples of conservative forces. This statement can be verified by calculating either $\nabla \times \boldsymbol{F} = 0$ or $\mathcal{W}_c = \oint \boldsymbol{F} \bullet d\boldsymbol{s} = 0$.

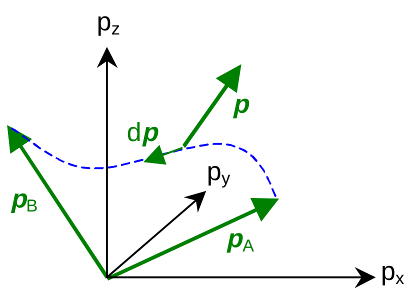
To reiterate, the work done by a conservative force is independent of the path taken, and is determined entirely by the coordinates of the initial and final locations. Or, (equivalently!) a conservative force is irrotational, has zero curl.

Can we ever directly calculate the work in a very general way for a force? The answer is yes, if the force is the total or net force acting on a particle. That is, if the force is that appearing in Newton's Second Law of Motion, $\boldsymbol{F}_N = \frac{d\boldsymbol{p}}{dt}$, where $\boldsymbol{p}$ is the momentum of the particle. Momentum is the product of the particle's mass and its velocity: $\boldsymbol{p} = m\boldsymbol{v}$.

$$\mathscr{W}_N = \int_A^B \boldsymbol{F}_N \bullet d\boldsymbol{s}$$



$$\mathscr{W}_N = \frac{1}{m}\int_{\boldsymbol{p}_A}^{\boldsymbol{p}_B} \boldsymbol{p} \bullet d\boldsymbol{p} = \frac{p_B^2}{2m} - \frac{p_B^2}{2m}$$



By a transformation of variable from that describing the path in real space, $d\boldsymbol{s}$, to that in momentum space, $d\boldsymbol{p}$, we can directly integrate the form for the work done by the net force. For the integrand,

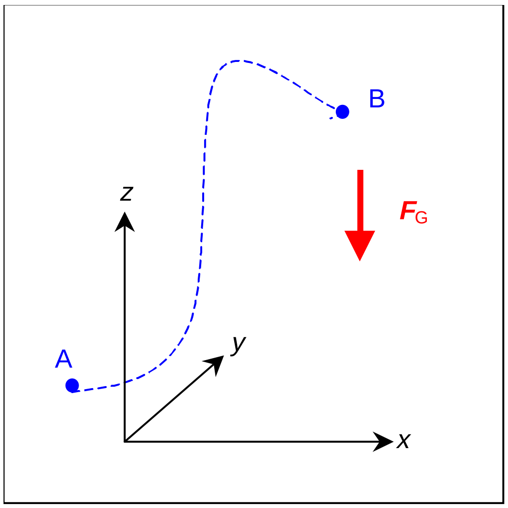
$$\boldsymbol{F}_N \bullet d\boldsymbol{s} = \frac{d\boldsymbol{p}}{dt} \bullet \boldsymbol{v}dt = \frac{d\boldsymbol{p}}{dt} \bullet \frac{\boldsymbol{p}}{m}dt = \frac{\boldsymbol{p} \bullet d\boldsymbol{p}}{m}.$$

Then

$$\mathscr{W}_N = \int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} \boldsymbol{F}_N \bullet d\boldsymbol{s} = \frac{1}{m}\int_{\boldsymbol{p}_A}^{\boldsymbol{p}_B} \boldsymbol{p} \bullet d\boldsymbol{p}$$

$$\mathscr{W}_N = \left.\frac{\boldsymbol{p} \bullet \boldsymbol{p}}{2m}\right|_{\boldsymbol{p}_A}^{\boldsymbol{p}_B} = \frac{p_B^2}{2m} - \frac{p_A^2}{2m}.$$

The quantity $\dfrac{p^2}{2m} = \tfrac{1}{2}mv^2$ is evidently pretty special, and so it is given a special name, the **kinetic energy**. And the remarkable result we have just obtained is called the **work-energy theorem**. Simply stated, this theorem asserts that, when any object's momentum is changed (that is, whenever it is accelerated) the work done by the net force acting on the object is the change in the object's kinetic energy.



Given a conservative force, then the work done by the force along a path depends only on the endpoints of the path. In such a circumstance (*only* if the force is conservative), a *Potential Energy*, U, can be associated with the conservative force, which by definition is always:

$$U_B - U_A \equiv -\mathcal{W}_{AB} = -\int_A^B \boldsymbol{F} \bullet d\boldsymbol{s}\,. \quad [2]$$

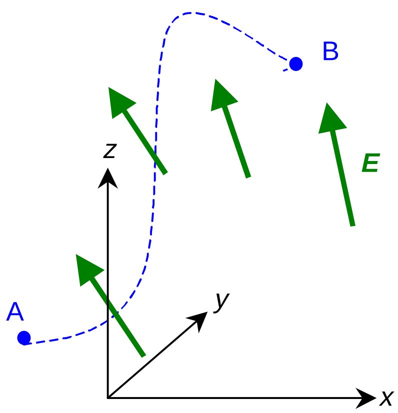
Exercise: Use this definition to show that the potential energy gained by moving a particle of mass m from a point $A\left(x_A, y_A, z_A\right)$ to point $B\left(x_B, y_B, z_B\right)$ in the presence of the local gravitational force $\boldsymbol{F}_g = mg\left(-\hat{\boldsymbol{k}}\right)$ is given by

$$U_B - U_A = mg\left(z_B - z_A\right).$$

For the electrostatic force, we define the electrostatic field $\boldsymbol{E}$ by, $\boldsymbol{F}_E = q_0\boldsymbol{E}$, where q_0 is the test charge that experiences the force. Putting this into the expression for the work ([2]), we have

$$U_B - U_A \equiv -\mathcal{W}_{AB} = -\int_A^B q_0\boldsymbol{E} \bullet d\boldsymbol{s} = -q_0\int_A^B \boldsymbol{E} \bullet d\boldsymbol{s}.$$

$$V_B - V_A = -\int_A^B \boldsymbol{E} \bullet d\boldsymbol{s}$$



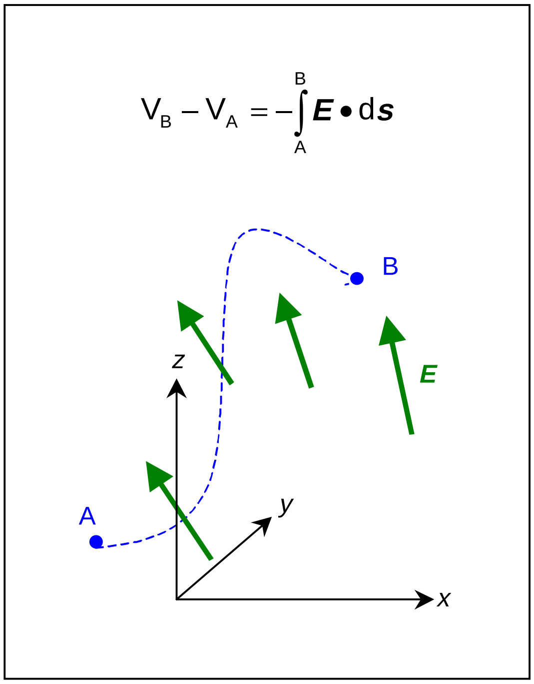
Note that, by dividing both sides of this equation by the test charge q_0, we have an equation for the potential energy per unit charge in the electric field between the points A and B. This quantity is called the *electrostatic potential difference*, or simply the *potential* between the points. We use the symbol V to represent the potential.

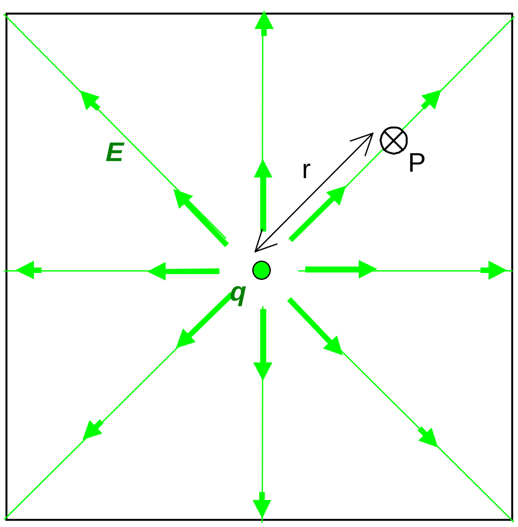
$$\frac{U_B - U_A}{q_0} \equiv V_B - V_A = -\int_A^B \boldsymbol{E} \bullet d\boldsymbol{s}\,.$$

We have used the precisely defined scientific terms for this quantity, and carefully defined it mathematically. But you should understand that this is exactly what is meant by the everyday term "the *voltage* between points A and B". You can see from the definition that the potential has units of the electric field times a length. The SI unit for the potential is the Volt, so

$1\,\text{Volt} = 1\frac{\text{N}\cdot\text{m}}{\text{C}} = 1\frac{\text{Newton}\cdot\text{meter}}{\text{Coulomb}}$. It is common to state the units of the electric field as $\frac{\text{Volt}}{\text{m}}$.

Next we explore the consequences of the definition of the potential as a path integral.





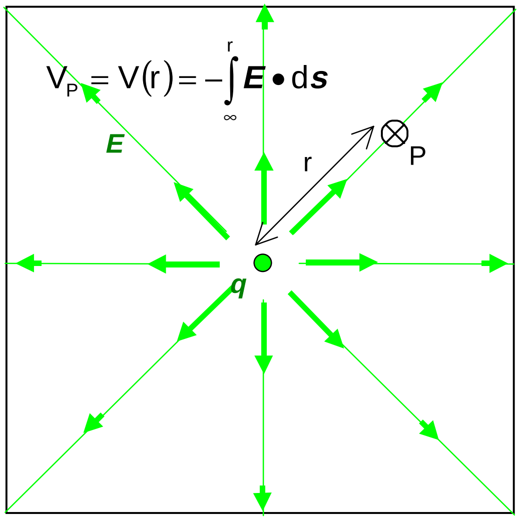
The first thing to note from the definition

$V_B - V_A = -\int_A^B \boldsymbol{E} \bullet d\boldsymbol{s}$ is that the point A

and the value of the potential at that point, V_A, are arbitrary. One is free to choose the point where the potential has the value zero, because only differences in the potential have physical consequence. The general rule is to pick a convenient reference point A, usually designating $V = 0$ at the reference point. Then the potential at any other point depends on the electric field all along the path from the reference point.

For distributions of charge, it is usually most convenient to choose the reference point to be infinitely removed from the distribution. We can see why by calculating the potential in the neighborhood of an isolated point charge using infinity as the reference point.

We choose an observation point P at some distance r away from the charge q. Then we choose a reference point infinitely removed from the charge: $V_\infty = 0$.

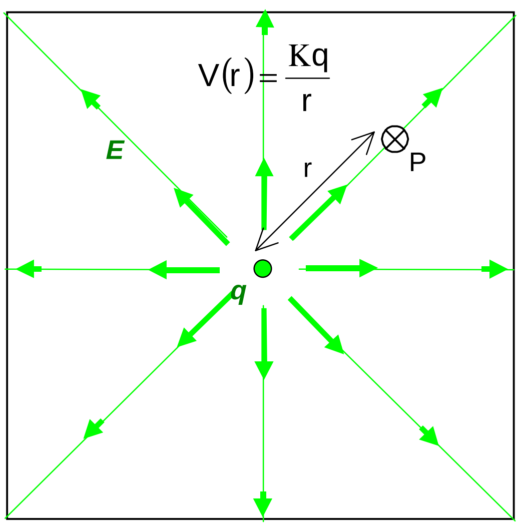


Now we calculate the potential at the observation point P from the definition.

$$V_P - V_\infty = V_P = -\int_{\infty}^{r} \boldsymbol{E} \bullet d\boldsymbol{s}.$$

Making things easy for ourselves, we choose a radial path inward from the reference point at infinity. In other words, we choose the path element $d\boldsymbol{s} = du\hat{\boldsymbol{u}}$, using u as a radial variable of integration. Then the integral becomes

$V_P = V(r) = -\int_{\infty}^{r} \frac{Kq}{u^2} \hat{\boldsymbol{u}} \bullet du\hat{\boldsymbol{u}}$, where we have inserted the Coulomb Law form for the electric field of the point charge.

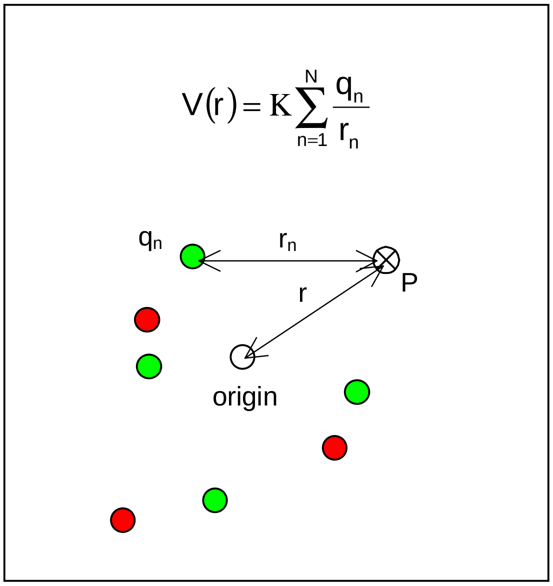


The integration has become simple, with the result:

$$V(r) = -Kq\int_{\infty}^{r} \frac{du}{u^2} = -Kq\left(\left.\frac{-1}{u}\right|_{\infty}^{r}\right) = \frac{Kq}{r}$$

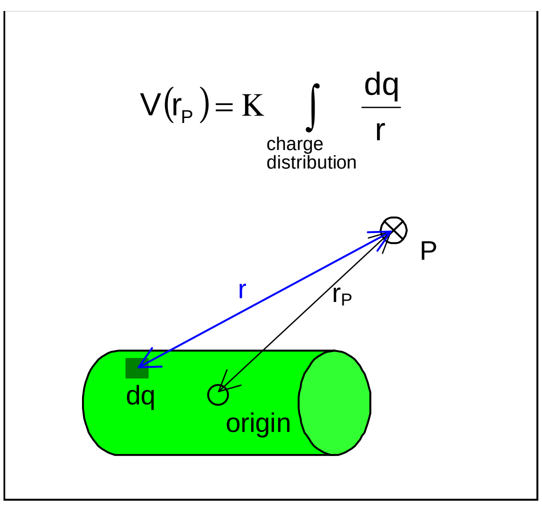
$$V(r) = \frac{Kq}{r}.$$

Here one can see why the choice of a reference point of zero potential at infinity is "convenient". That is the choice that results in the simplest possible form for the potential of the isolated point charge.



The result for the potential of an isolated charge provides a way of calculating the potential of a charge distribution when we do not know the electric field associated with it. Because electric fields can be linearly superposed, so can the potentials defined by a path integration of the fields. We can simply add up the contributions.

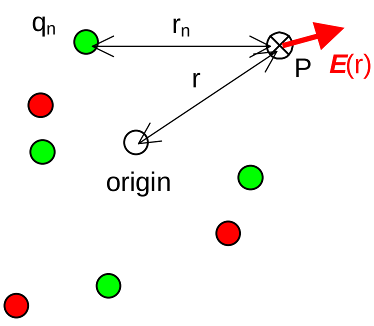
For a charge distribution consisting of N discrete point charges $q_1, \ldots, q_N$ at distances $r_1, \ldots, r_N$, respectively, from the observation point: $V(r) = K\sum_{n=1}^{N}\frac{q_n}{r_n}$.



For a finite extended charged object, the potential at the observation point P is given by the superposition of the potentials of all charge elements in the charge distribution:

$$V(r_P) = K\int_{\substack{\text{charge}\\\text{distribution}}}\frac{dq}{r}.$$

$$\boldsymbol{E}(\mathrm{r}) = \mathrm{K}\sum_{n=1}^{N}\frac{q_n}{r_n^2}\hat{\boldsymbol{r}}_n$$



Note that these superposition expressions are similar to the Coulomb Law expressions for the electric field: $\boldsymbol{E}(\mathrm{r}) = \mathrm{K}\sum_{n=1}^{N}\frac{q_n}{r_n^2}\hat{\boldsymbol{r}}_n$ and

$$\boldsymbol{E}(r_P) = \mathrm{K}\int_{\substack{\text{charge}\\ \text{distribution}}}\frac{dq\,\hat{\boldsymbol{r}}}{r^2}$$, respectively.

Usually, the calculation of the potential for a charge distribution is easier than calculating the electric field by the Coulomb Law, primarily because the potential is a scalar quantity, rather than a vector.

Can we determine the electric field if the potential is known? The answer is "yes", and it is easily done.

$$V(x,y,z) - V_R = -\int_R^P \boldsymbol{E} \bullet d\boldsymbol{s}$$

$$V - V_R = -\int_R^P \left(E_x dx + E_y dy + E_z dz\right)$$

$$\frac{\partial V}{\partial x} = -E_x\,,\quad \frac{\partial V}{\partial y} = -E_y\,,\quad \frac{\partial V}{\partial z} = -E_z$$

$$\nabla V = \frac{\partial V}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial V}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial V}{\partial z}\hat{\boldsymbol{k}}$$

$$\nabla V = -E_x\hat{\boldsymbol{i}} - E_y\hat{\boldsymbol{j}} - E_z\hat{\boldsymbol{k}}$$

$$\boldsymbol{E} = -\nabla V$$

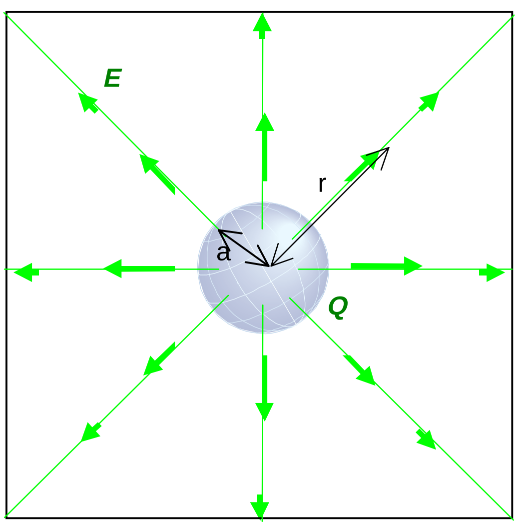
Writing the element of the path as

$d\boldsymbol{s} = dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}}$, we can expand the dot product under the integral, and see that the partial derivatives of the potential are the negative of the components of the electric field. (Remember that the value of the reference potential V_R is a constant, usually zero.) That relationship can be compactly expressed using the Del operator as

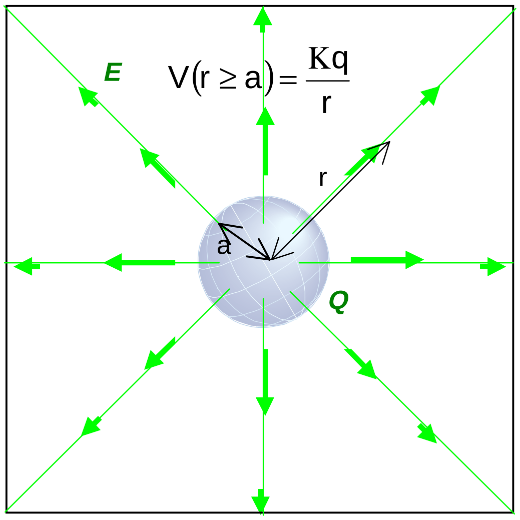
$\boldsymbol{E} = -\nabla V$. The expression ∇V is called the *gradient* of the potential V.

In most cases, the easiest procedure to find the electric field for a charge distribution is to first calculate the potential, then take its gradient to get the electric field.

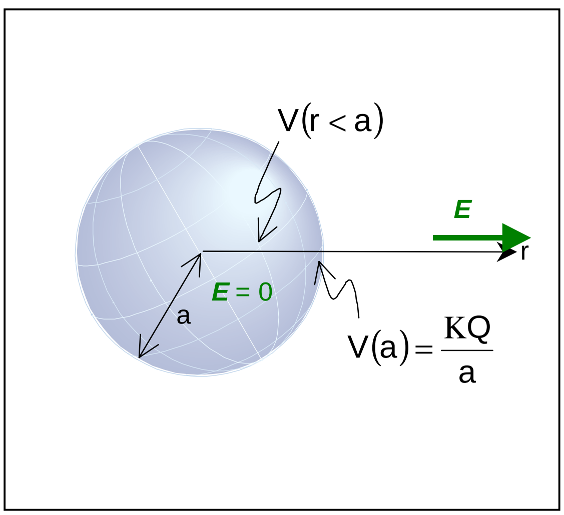
Finally, we note a couple of properties that the potential has by virtue of its definition as a path integral: 1) The potential must be *continuous*, and 2) the potential will generally have non-zero value even in regions of space for which the electric field is zero.



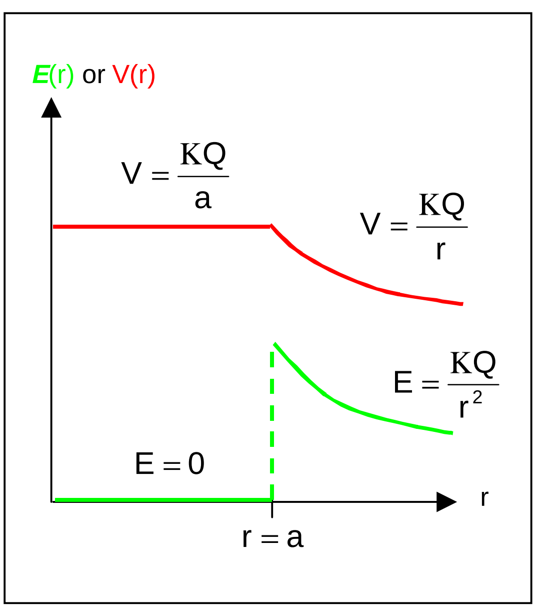
To illustrate these properties by an example, consider a charged conducting sphere of radius a. The entire charge Q of this object lies on the surface of the sphere, in the form of a constant surface charge density $\sigma_0 = \dfrac{Q}{4\pi a^2}$. The electric field (perhaps calculated using the Gauss Law) is given by $\boldsymbol{E} = \begin{cases} \dfrac{KQ}{r^2}\hat{\boldsymbol{r}}, r \geq a \\ 0, r < a \end{cases}$. The electric field arises from the surface charge, and is zero inside the surface of the sphere.



In the region outside the charged sphere $(r \geq a)$, the calculation for the potential is the same as was done above for the point charge, yielding $V(r \geq a) = \frac{KQ}{r}$. In particular, the value of the potential at the surface $(r = a)$ arises from all of the electric field encountered in the path from the reference point at infinity, and has the value $V(a) = \frac{KQ}{a}$.



The potential inside the sphere $V(r < a)$ also depends on all the electric field encountered on the path from the reference point, even though the electric field in this region is zero.



Specifically,

$$V(r < a) = -\int_{\infty}^{r} \boldsymbol{E} \bullet d\boldsymbol{s}$$

$$V(r < a) = -\int_{\infty}^{a} \boldsymbol{E}(r \geq a) \bullet d\boldsymbol{s} - \int_{a}^{r} \boldsymbol{E}(r < a) \bullet d\boldsymbol{s}$$

$$V(r < a) = V(a) - \int_{a}^{r} 0 \bullet d\boldsymbol{s} = V(a).$$

Inside the sphere, where the electric field is zero, the potential remains constant at the value obtained at the surface of the sphere.

Instructor's Guide

This project is most appropriate for students who have completed at least two semesters of calculus study and are currently enrolled in the second semester of the introductory physics series on electricity and magnetism. However, it could also be useful for rather advanced students taking the first physics course on mechanics, or as "background repair" for struggling students in a junior-level electrodynamics course.

Ideally, the project would be a joint effort by a team of two students, their physics instructor, and a mathematics instructor with experience in teaching calculus. (A single student could also undertake the project, but the advantages of peer teaching would be lost.) The project begins with a kickoff meeting of all participants, in which the students receive a copy of the storyboard and the activities, and the instructors review the storyboard in some detail. This review should stress the unity of concepts connecting work, conservative forces and the electrostatic potential. The students leave the first meeting with the task of beginning work together on the problem set, and with a scheduled next meeting with one or both of the instructors. A time limit for completion of the project should be set at the first meeting, and should be short enough to require rather intensive effort by the students. A time limit of two weeks appears appropriate for average students, although exceptional ones might be prepared to complete it in as little as one week.

The activities are challenging to all but the most exceptional students, so the students should frequently meet with one or both instructors during the project for coaching. The coaching meetings should occur every couple of days, and can be expected to last about one-half hour each. Some of the coaching will deal with technical details such as techniques of integration and differentiation. Still, the instructors should repeatedly refer back to the concepts presented in the storyboard as the basis for solving the problems.

Our first experience with use of this ILAP occurred during the Fall Semester of 1998, with participation by ten students (of a class of nineteen students) who were taking the introductory electricity and magnetism course at Clark Atlanta University. The inducement was that their grade on the ILAP could replace an existing test grade in the course. The participants formed five teams of two students each. The results were very positive for those students who actively participated, as discussed below.

Of the five teams, two did not seriously apply themselves to the ILAP, received poor grades for it, and showed no measurable benefit in subsequent tests given during the semester. The remaining three teams worked diligently on the activities, used their physics and mathematics instructors appropriately as coaches, and received good grades on the ILAP. On one team, the two students participated as equals, while the other two of the three teams had one identifiable leader who contributed most of the effort, with the other member of the team less active. The six students on these three teams unanimously expressed the opinion that the storyboard presentation was useful in learning the material. More importantly, the four students who were fully active participants all displayed *full comprehension* of the use of path integrals in subsequent tests given during the semester, gaining full credit for all questions posed to them involving calculations with path integrals. This result was a remarkable improvement for all four of these students.

Therefore, we intend to continue use of the ILAP in subsequent semesters.

Instructor's Solutions to Activities

1. (a) It is instructive to first note that the force is constant, unvarying over any path between the points A and B, and therefore that both the force and the dot product operation can be taken outside the integral. $\mathcal{W}_g = \int_A^B \boldsymbol{F}_g \bullet d\boldsymbol{s} = \boldsymbol{F}_g \bullet \int_A^B d\boldsymbol{s}$. The remaining integral is simply the vector from the initial point A to the final point B,
$\int_A^B d\boldsymbol{s} = \boldsymbol{s} = (x_B - x_a)\hat{\boldsymbol{i}} + (y_B - y_a)\hat{\boldsymbol{j}} + (z_B - z_a)\hat{\boldsymbol{k}}$, which in turn depends only on the coordinates of the initial and final points. Thus $\mathcal{W}_g = \boldsymbol{F}_g \bullet \int_A^B d\boldsymbol{s} = \boldsymbol{F}_g \bullet \boldsymbol{s}$ is conservative, and the same argument holds for *any constant force*. QED.

   One can solve this problem very specifically by simply substituting the forms $\boldsymbol{F}_g = mg(-\hat{\boldsymbol{k}})$ and $d\boldsymbol{s} = dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}}$ into the definition of the work and performing the operations, to wit: $\mathcal{W}_g = \int_A^B \boldsymbol{F}_g \bullet d\boldsymbol{s} = \int_{x_A, y_A, z_A}^{x_B, y_B, z_B} \left[mg(-\hat{\boldsymbol{k}}) \bullet (dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}})\right] = \int_{z_A}^{z_B} (-mg)dz = -mg(z_B - z_A)$.
Of course, the same answer is generated by either method.
$\mathcal{W}_g = \boldsymbol{F}_g \bullet \boldsymbol{s} = \left[mg(-\hat{\boldsymbol{k}})\right] \bullet \left[(x_B - x_a)\hat{\boldsymbol{i}} + (y_B - y_a)\hat{\boldsymbol{j}} + (z_B - z_a)\hat{\boldsymbol{k}}\right] = -mg(z_B - z_a)$.

   (b) The element of path length for the circle given in the problem is
$d\boldsymbol{s} = Rd\theta\hat{\boldsymbol{e}} = R(\cos\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{k}})d\theta$. Substitution into the form for the work yields
$\mathcal{W}_g = \oint \boldsymbol{F}_g \bullet d\boldsymbol{s} = \int_0^{2\pi} mg(-\hat{\boldsymbol{k}}) \bullet R(\cos\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{k}})d\theta = -mgR\int_0^{2\pi} \sin\theta d\theta = mgR(\cos\theta|_0^{2\pi} = 0$
QED.
(c) $\nabla \times \boldsymbol{F}_g = \left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right) \times \left[mg(-\hat{\boldsymbol{k}})\right] = \hat{\boldsymbol{j}}\frac{\partial}{\partial x}(mg) + (-\hat{\boldsymbol{i}})\frac{\partial}{\partial y}(mg) = 0 + 0 = 0$. QED.

2. (a) Choose a coordinate system in the plane of the circular orbit, with the origin at the center of the earth. The satellite's orbital path is then a circle of radius r ($r > R_E$, where R_E is the radius of the earth), and an element of the path is $d\boldsymbol{s} = rd\phi\hat{\boldsymbol{\phi}}$. Then the work done by gravity in any part of the orbit is $\mathcal{W}_G = \int_{\phi_A}^{\phi_B} \boldsymbol{F}_G \bullet d\boldsymbol{s} = \int_{\phi_A}^{\phi_B} G\frac{m_E m}{r^2}(-\hat{\boldsymbol{r}}) \bullet \hat{\boldsymbol{\phi}} rd\phi = 0$, because the dot product is identically zero. ($\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are orthogonal unit vectors.) QED. Note that if the orbit is not perfectly circular but is elliptical, then $d\boldsymbol{s}$ has components along both $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$. In this case the work done by gravity in a segment of the orbit may be either positive (the satellite is approaching the earth) or negative (the satellite is receding from the earth). Still, the work done for a complete orbit is easily shown to be zero (the angular integration vanishes because the dot product is zero and the radial integration vanishes because the lower and upper limits are identical).

(b) Note that $\boldsymbol{F}_G$ has the form $\boldsymbol{F}_G = F_{Gr}(r)\hat{\boldsymbol{r}}$. All the angular components and angular derivatives are zero: $F_{G\theta} = F_{G\phi} = 0$ and $\frac{\partial F_{Gr}}{\partial\theta} = \frac{\partial F_{Gr}}{\partial\phi} = 0$. So each one of the six terms in the spherical coordinates expression for $\nabla\times\boldsymbol{F}_G$ is zero. QED.

3. (a) This is only a slight generalization of 2(a). The dot product eliminates all but the radial dependence of the closed path integral.
$\oint \boldsymbol{E}\bullet d\boldsymbol{s} = \oint \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}\bullet\left(dr\hat{\boldsymbol{r}} + r\,sin\theta d\phi\hat{\boldsymbol{\phi}} + rd\theta\hat{\boldsymbol{\theta}}\right) = \frac{q}{4\pi\varepsilon_0}\oint\frac{dr}{r^2}$. Denoting by r_A the starting and ending radial distances for the closed path,
$\oint \boldsymbol{E}\bullet d\boldsymbol{s} = \frac{q}{4\pi\varepsilon_0}\int_{r_A}^{r_A}\frac{dr}{r^2} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r}\bigg|_{r_a}^{r_a}\right) = 0$. QED.

   (b) This is done exactly as 2(b). Replace $\boldsymbol{F}_G$ with $\boldsymbol{E}$ and F_G with E_G in the answer to 2(b) to show that $\nabla\times\boldsymbol{E} = 0$.

4. The closed path indicated consists of a horizontal line segment plus a semicircular line segment.
$$\mathcal{W}_f = \oint \boldsymbol{f}_k\bullet d\boldsymbol{s} = \int_{-R}^{R}\mu_k mg\left(-\hat{\boldsymbol{i}}\right)\bullet dx\hat{\boldsymbol{i}} + \int_0^{\pi}\mu_k mg\left(-\hat{\boldsymbol{\theta}}\right)\bullet Rd\theta\hat{\boldsymbol{\theta}}$$
. QED.
$$\mathcal{W}_f = -\mu_k mg(2R+\pi R) = -\mu_k mgR(2+\pi) \neq 0$$
Addressing the curl of the frictional force, we separately consider the straight and semicircular path segments. Straightforwardly applying the definition of the curl, we find for the straight segment $\nabla\times\boldsymbol{f}_k = \left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right)\times\left[\mu_k mg\left(-\hat{\boldsymbol{i}}\right)\right] = 0$.* And for the semicircular segment:
$$\nabla\times\boldsymbol{f}_k = \left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right)\times\left[\mu_k mg\left(sin\theta\hat{\boldsymbol{i}} - cos\theta\hat{\boldsymbol{j}}\right)\right]$$
$$\nabla\times\boldsymbol{f}_k = \frac{\mu_k mg}{R}\left[\left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right)\times\left(y\hat{\boldsymbol{i}} - x\hat{\boldsymbol{j}}\right)\right] = \frac{2\mu_k mg}{R}\left(-\hat{\boldsymbol{k}}\right) \neq 0$$
, where we have used the replacements $x = R\,cos\theta$ and $y = R\,sin\theta$.
* The perceptive student may object "If the object returns to the initial point backward along the same straight line, the path integral remains nonzero, but the curl appears to be zero." From the physicist's point of view, this can be dismissed by the impossibility of exactly retracing a mathematical line with a physical object, and any separation of the return path from the original generates a nonzero curl, that is, a variation in y of the unit vector $\left(-\hat{\boldsymbol{v}}\right)$. From the mathematical point of view, the objection might lead to an interesting discussion of limit theorems and continuity and differentiability of functions. Such a discussion is beyond the scope of this ILAP.

5. (a) Throughout this solution, the dummy radial variable u is used in the integrand to avoid confusion with the all-important upper limit of the path integral. The radial path inward from infinity is used, so that $d\boldsymbol{s} = du\hat{\boldsymbol{u}}$. Because the path integral must begin at the reference point at $r = \infty$, we first address the part of the domain outside the charge distribution $(r > 3a)$ and use the form of the electric field that applies in this region:

$$V(r>3a) = -\int_{\infty}^{r} \boldsymbol{E}(u>3a)\bullet d\boldsymbol{s} = -\int_{\infty}^{r} \frac{q}{2\pi\varepsilon_0 u^2}(-\hat{\boldsymbol{u}})\bullet du\,\hat{\boldsymbol{u}} = \frac{q}{2\pi\varepsilon_0}\int_{\infty}^{r}\frac{du}{u^2} = \frac{-q}{2\pi\varepsilon_0 r}.$$

The limit value at the inward extreme of this region is $V(3a) = \frac{-q}{6\pi\varepsilon_0 a}$.

Continuing the path integral into the region for which $2a \le r \le 3a$, we have

$$V(2a \le r \le 3a) = -\int_{\infty}^{r} \boldsymbol{E}\bullet d\boldsymbol{s} = -\int_{\infty}^{3a} \boldsymbol{E}(u>3a)\bullet d\boldsymbol{s} - \int_{3a}^{r} \boldsymbol{E}(2a \le u \le 3a)\bullet d\boldsymbol{s}$$

$$V(2a \le r \le 3a) = V(3a) - \int_{3a}^{r} 0\bullet d\boldsymbol{s} = V(3a).$$

The potential is constant in this region. The inner limit value is $V(2a) = V(3a) = \frac{-q}{6\pi\varepsilon_0 a}$.

For the remainder of the solution, we take as given the results of previous integrations. Continuing inward into the region $a \le r \le 2a$, we have

$$V(a \le r \le 2a) = V(2a) - \int_{2a}^{r} \boldsymbol{E}(a \le r \le 2a)\bullet d\boldsymbol{s} = V(2a) - \int_{2a}^{r}\frac{q}{4\pi\varepsilon_0 u^2}\hat{\boldsymbol{u}}\bullet du\,\hat{\boldsymbol{u}}$$

$$V(a \le r \le 2a) = V(2a) - \frac{q}{4\pi\varepsilon_0}\int_{2a}^{r}\frac{du}{u^2} = V(2a) + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{u}\Big|_{2a}^{r}\right)$$

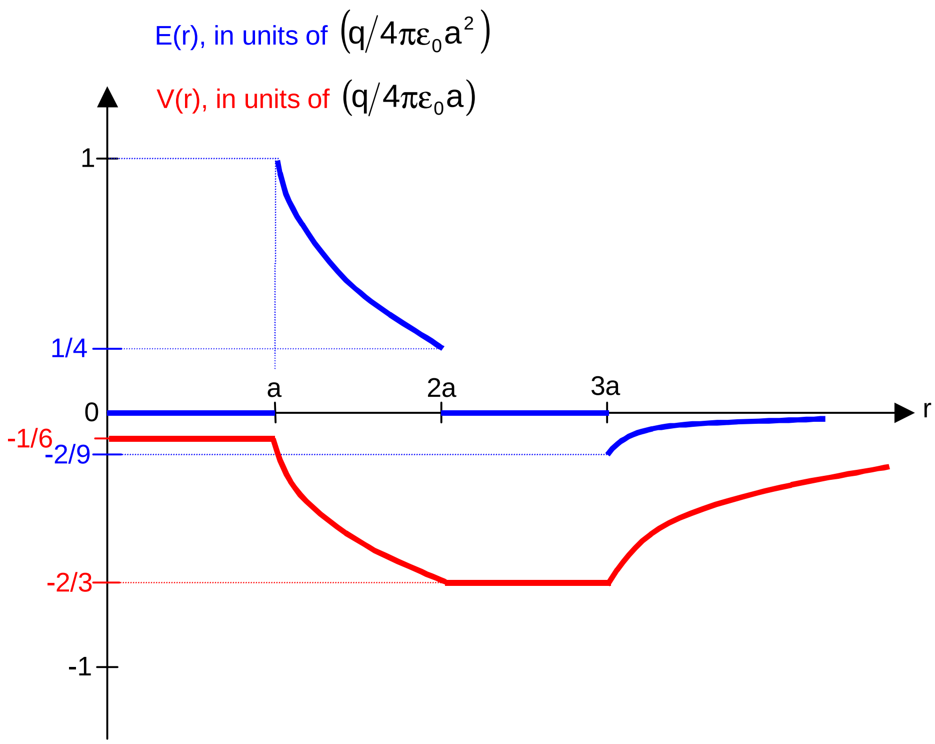
$$V(a \le r \le 2a) = \frac{-q}{6\pi\varepsilon_0 a} + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{2a}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{7}{6a}\right).$$

The inward limit value in this region is $V(a) = \frac{-q}{24\pi\varepsilon_0 a}$.

Finally, in the region $r < a$, we have

$V(r<a) = V(a) - \int_{a}^{r} \boldsymbol{E}(r<a)\bullet d\boldsymbol{s} = V(a) - \int_{a}^{r} 0\bullet du\,\hat{\boldsymbol{u}} = V(a)$. As in the other region for which the electric field is zero, the potential remains constant at its value on the boundary throughout the region. The inward limit value is $V(0) = V(a) = \frac{-q}{24\pi\varepsilon_0 a}$.

(b) In plotting the electric field and the potential, we display the electric field as a heavy blue curve and the potential as a heavy red curve. The electric field is plotted in units of $\frac{q}{4\pi\varepsilon_0 a^2}$, and the potential in units of $\frac{q}{4\pi\varepsilon_0 a}$. These units arise naturally in terms of known quantities in the solution. This type of simultaneous graphing can provide the student with a deeper understanding of the relationship between the electric field and the potential. The electric field has discontinuities at the charged surfaces of the conductors. The potential must be continuous, but exhibits discontinuities in slope at the charged surfaces. The electric field is everywhere equal to the negative slope of the potential, in accordance with the relationship $\boldsymbol{E} = -\nabla V$.



6. (a) As in Activity 5, the dummy radial variable u is used in all integrands, to avoid confusion with the upper limit of the path integral. Within the outer conductor $b < r < c$ the electric field is zero, and the potential remains constant at the zero value of the reference point at $r = c$.

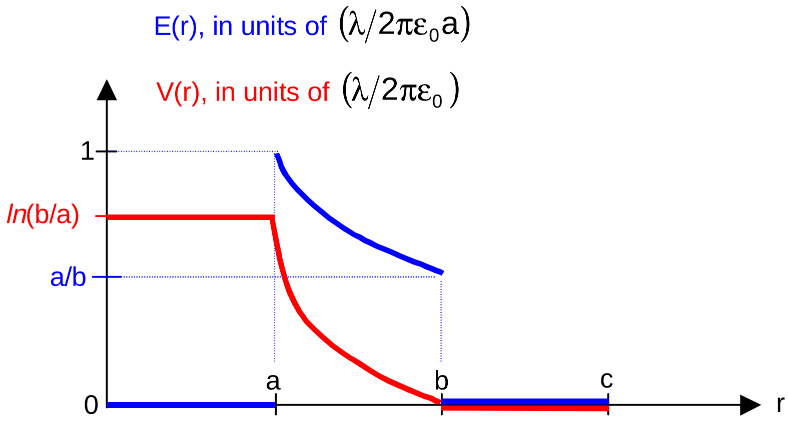
$V(b<r<c) = -\int_c^r \boldsymbol{E}(b<u<c)\bullet d\boldsymbol{s} = -\int_c^r 0 \bullet du\hat{\boldsymbol{u}} = 0$. The inner limit value in this region is $V(b) = 0$. Between the conductors $a \le r \le b$ and the potential is given by

$$V(a \le r \le b) = V(b) - \int_b^r \boldsymbol{E}(a \le r \le b)\bullet d\boldsymbol{s} = -\int_b^r \frac{\lambda}{2\pi\varepsilon_0 u}\hat{\boldsymbol{u}} \bullet du\hat{\boldsymbol{u}} = -\frac{\lambda}{2\pi\varepsilon_0}\int_b^r \frac{du}{u} = \frac{\lambda}{2\pi\varepsilon_0} \mathit{ln}\left(\frac{b}{r}\right).$$

The inner limit value in this region is $V(a) = \frac{\lambda}{2\pi\varepsilon_0}\mathit{ln}\left(\frac{b}{a}\right)$. Within the inner conductor $r < a$ the electric field is again zero, and the potential remains constant at the value $V(a)$.

$V(r<a) = V(a) - \int_a^r \boldsymbol{E}(r<a)\bullet d\boldsymbol{s} = V(a) - \int_a^r 0 \bullet du\hat{\boldsymbol{u}} = V(a) = \frac{\lambda}{2\pi\varepsilon_0}\mathit{ln}\left(\frac{b}{a}\right)$. The inner limit value in this region is $V(0) = V(a) = \frac{\lambda}{2\pi\varepsilon_0}\mathit{ln}\left(\frac{b}{a}\right)$.

(b) The graph is similar to that for Activity 5. In this graph the electric field is plotted in units of $\frac{\lambda}{2\pi\varepsilon_0 a}$, and the potential in units of $\frac{\lambda}{2\pi\varepsilon_0}$. The curves as drawn correspond to $\frac{b}{a} \approx 2$.



7. (a) The distances from each charge to the observation point can be found in a number of ways, by vector analysis, geometrical construction of right triangles, or by use of the law of cosines. Using vector analysis here, the vector from the positive charge to the observation point is $\boldsymbol{r}_+ = u\hat{\boldsymbol{u}} + (z-a)\hat{\boldsymbol{k}}$ and its length is

$$r_+ = (\boldsymbol{r}_+ \bullet \boldsymbol{r}_+)^{1/2} = \left[u^2 + (z-a)^2\right]^{1/2} = \left[r^2 \sin^2\theta + (r\cos\theta - a)^2\right]^{1/2}$$

$$r_+ = \left(r^2 + a^2 - 2ar\cos\theta\right)^{1/2}.$$

Similarly, $r_- = \left[u^2 + (z+a)^2\right]^{1/2} = \left(r^2 + a^2 + 2ar\cos\theta\right)^{1/2}$. Substitution of these distances into the form for the potential gives the desired result

$$V(r,\theta) = K\sum_{n=1}^{2}\frac{q_n}{r_n} = K\left[\frac{q}{r_+} + \frac{-q}{r_-}\right] = Kq\left[\left(r^2 + a^2 - 2ar\cos\theta\right)^{-1/2} - \left(r^2 + a^2 + 2ar\cos\theta\right)^{-1/2}\right]$$

QED.

(b) Each term in the potential vanishes in the limit of infinite distance of the observation point. $\lim_{r\to\infty}\left(r^2 + a^2 \pm 2ar\cos\theta\right)^{-1/2} = 0$, so $\lim_{r\to\infty}\left[V(r,\theta)\right] = 0$. QED.

(c) Substituting $\theta \to (\pi - \theta)$ in the form for the potential yields

$$V(r,\pi-\theta) = Kq\left[\left(r^2 + a^2 - 2ar\cos\{\pi-\theta\}\right)^{-1/2} - \left(r^2 + a^2 + 2ar\cos\{\pi-\theta\}\right)^{-1/2}\right]$$

$$V(r,\pi-\theta) = Kq\left[\left(r^2 + a^2 + 2ar\cos\theta\right)^{-1/2} - \left(r^2 + a^2 - 2ar\cos\theta\right)^{-1/2}\right] = -V(r,\theta).$$

QED. Note use of the trigonometric identity $\cos(\pi-\theta) = -\cos\theta$.

(d) Given the form for the gradient in spherical coordinates and

$$\boldsymbol{E}(r,\theta) = -\nabla V(r,\theta) = -\left[\frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{\phi}}\right]$$, we first note that the potential is independent of the angle ϕ (reflecting the cylindrical symmetry of the potential about the *z*-axis), so that the third term in the gradient vanishes. That is, $E_\phi = 0$, the electric field also

has cylindrical symmetry about the *z*-axis. Writing $\boldsymbol{E}(r,\theta)=E_r\hat{\boldsymbol{r}}+E_\theta\hat{\boldsymbol{e}}$, we calculate each component separately.

$$E_r=-\frac{\partial V}{\partial r}=-Kq\frac{\partial}{\partial r}\left[\left(r^2+a^2-2ar\cos\theta\right)^{-1/2}-\left(r^2+a^2+2ar\cos\theta\right)^{-1/2}\right]$$

$$E_r=-Kq\left[\frac{\left(-\frac{1}{2}\right)(2r-2a\cos\theta)}{\left(r^2+a^2-2ar\cos\theta\right)^{3/2}}-\frac{\left(-\frac{1}{2}\right)(2r+2a\cos\theta)}{\left(r^2+a^2+2ar\cos\theta\right)^{3/2}}\right]$$

$$E_r=Kq\left[\frac{(r-a\cos\theta)}{\left(r^2+a^2-2ar\cos\theta\right)^{3/2}}-\frac{(r+a\cos\theta)}{\left(r^2+a^2+2ar\cos\theta\right)^{3/2}}\right].$$

Comparison reveals agreement with the radial component stated in Activity 7(d). For the azimuthal component:

$$E_\theta=-\left(\frac{1}{r}\right)\frac{\partial V}{\partial\theta}=\frac{-Kq}{r}\frac{\partial}{\partial\theta}\left[\left(r^2+a^2-2ar\cos\theta\right)^{-1/2}-\left(r^2+a^2+2ar\cos\theta\right)^{-1/2}\right]$$

$$E_\theta=\frac{-Kq}{r}\left[\frac{\left(-\frac{1}{2}\right)(2ar\sin\theta)}{\left(r^2+a^2-2ar\cos\theta\right)^{3/2}}-\frac{\left(-\frac{1}{2}\right)(-2ar\sin\theta)}{\left(r^2+a^2+2ar\cos\theta\right)^{3/2}}\right]$$

$$E_\theta=Kqa\sin\theta\left[\frac{1}{\left(r^2+a^2-2ar\cos\theta\right)^{3/2}}+\frac{1}{\left(r^2+a^2+2ar\cos\theta\right)^{3/2}}\right].$$

Again, comparison reveals agreement with the azimuthal component stated in Activity 7(d).

(e) Note that E_r is a difference of two terms, and E_θ is a sum of two terms. For each of the components, the transformation of $\theta\to\pi-\theta$ exchanges the identities of the terms. This leaves the sum in E_θ unchanged, but reverses the sign of the difference in E_r. Thus if $\boldsymbol{E}(r,\theta)=E_r\hat{\boldsymbol{r}}+E_\theta\hat{\boldsymbol{e}}$, then $\boldsymbol{E}(r,\pi-\theta)=-E_r\hat{\boldsymbol{r}}+E_\theta\hat{\boldsymbol{e}}$. In the midplane $V(r,\frac{\pi}{2})=Kq\left[\left(r^2+a^2\right)^{-1/2}-\left(r^2+a^2\right)^{-1/2}\right]=0$ and the potential in this plane has no variation with radial distance. However, the gradient of the potential is perpendicular to the midplane, and so is the electric field. Note that

$E_r(r,\frac{\pi}{2})=Kq\left[\frac{r}{\left(r^2+a^2\right)^{3/2}}-\frac{r}{\left(r^2+a^2\right)^{3/2}}\right]=0$, while $E_\theta=\frac{2Kqa}{\left(r^2+a^2\right)^{3/2}}$. Because at $\theta=\frac{\pi}{2}$, $\hat{\boldsymbol{e}}=-\hat{\boldsymbol{k}}$, and $\boldsymbol{E}_{\text{midplane}}=\boldsymbol{E}(r)=\frac{2Kqa}{\left(r^2+a^2\right)^{3/2}}\left(-\hat{\boldsymbol{k}}\right)$. This result decreases monotonically with distance from the origin. The maximum value is $\boldsymbol{E}(0)=\frac{2Kq}{a^2}\left(-\hat{\boldsymbol{k}}\right)$.

(f) The path element for the path described is $d\boldsymbol{s}=d\rho\,\hat{\boldsymbol{n}}$ (here using ρ as the dummy radial variable in the integrand), and the equation for the potential in terms of the electric field is

$$V(r,\theta) = -\int_{\infty}^{r} \boldsymbol{E} \bullet d\boldsymbol{s} = -\int_{\infty}^{r} \left(E_\rho \boldsymbol{\tilde{n}} + E_\theta \boldsymbol{\hat{e}}\right) \bullet d\rho \boldsymbol{\tilde{n}} = -\int_{\infty}^{r} E_\rho d\rho$$

$$V(r,\theta) = -Kq \int_{\infty}^{r} \left[\frac{(\rho - a\cos\theta)}{\left(\rho^2 + a^2 - 2a\rho\cos\theta\right)^{3/2}} - \frac{(\rho + a\cos\theta)}{\left(\rho^2 + a^2 + 2a\rho\cos\theta\right)^{3/2}} \right] d\rho$$

$$V(r,\theta) = Kq \left\{ \left(\rho^2 + a^2 - 2a\rho\cos\theta\right)^{-1/2} - \left(\rho^2 + a^2 + 2a\rho\cos\theta\right)^{-1/2} \Big|_{\infty}^{r} \right.$$

$$V(r,\theta) = Kq \left[\left(r^2 + a^2 - 2ar\cos\theta\right)^{-1/2} - \left(r^2 + a^2 + 2ar\cos\theta\right)^{-1/2} \right].$$

QED.

(g) Taking limits by substitution in the forms for the potential and the electric field for the dipole simply yields the uninteresting result of zero. We look for forms describing the approximate behavior of the functions when $r >> a$. Note that the operation described by $\lim_{r>>a}$ is equivalent to $\lim_{(a/r)\to 0}$. This suggests factoring r from the denominators, which is legal because we are certainly far from $r = 0$, and which makes explicit the role of the small quantity (a/r). We can then use the binomial approximation to expand the radicals. Following this plan, we write

$$V(r,\theta) = Kq \left[\left(r^2 + a^2 - 2ar\cos\theta\right)^{-1/2} - \left(r^2 + a^2 + 2ar\cos\theta\right)^{-1/2} \right]$$

$$\tilde{V} = \lim_{r>>a} V(r,\theta) = \frac{Kq}{r} \lim_{\frac{a}{r}\to 0} \left[\left(1 + \left\{\frac{a}{r}\right\}^2 - 2\left\{\frac{a}{r}\right\}\cos\theta\right)^{-1/2} - \left(1 + \left\{\frac{a}{r}\right\}^2 + 2\left\{\frac{a}{r}\right\}\cos\theta\right)^{-1/2} \right].$$

The terms in $(a/r)^2$ can be neglected in comparison to those in (a/r), and the binomial approximation applied to take the limit, using $\delta = (\pm)2\left\{\frac{a}{r}\right\}\cos\theta$ and $n = -\frac{1}{2}$. (The minus sign in δ is used for the first term, the plus sign in the second term.) This procedure yields

$$\tilde{V} = \frac{Kq}{r} \left[\left(1 + \left(-\tfrac{1}{2}\right)\left[-2\left\{\frac{a}{r}\right\}\cos\theta\right]\right) - \left(1 + \left(-\tfrac{1}{2}\right)\left[2\left\{\frac{a}{r}\right\}\cos\theta\right]\right) \right]$$

$$\tilde{V} = \frac{Kq}{r} \left[\left(1 + \frac{a}{r}\cos\theta\right) - \left(1 - \frac{a}{r}\cos\theta\right) \right] = \frac{2Kqa\cos\theta}{r^2}.$$

Applying the same procedure to each component of the electric field, we find

$$\tilde{E}_r = \frac{Kq}{r^2} \lim_{\frac{a}{r}\to 0} \left[\frac{\left(1 - \frac{a}{r}\cos\theta\right)}{\left(1 + \frac{a^2}{r^2} - 2\frac{a}{r}\cos\theta\right)^{3/2}} - \frac{\left(1 + \frac{a}{r}\cos\theta\right)}{\left(1 + \frac{a^2}{r^2} + 2\frac{a}{r}\cos\theta\right)^{3/2}} \right].$$ Here we can neglect the second term in each numerator with respect to the first, and again neglect the squared terms in the denominators. Using the binomial approximation, we have

$$\tilde{E}_r = \frac{Kq}{r^2} \left[\left(1 + 3\frac{a}{r}\cos\theta\right) - \left(1 - 3\frac{a}{r}\cos\theta\right) \right] = \frac{6Kqa\cos\theta}{r^3}$$ for the radial component. For the azimuthal component

$$\tilde{E}_\theta = \frac{Kqa\,sin\,\theta}{r^3}\lim_{\frac{a}{r}\to 0}\left[\frac{1}{\left(1+\frac{a^2}{r^2}-2\frac{a}{r}cos\,\theta\right)^{3/2}}+\frac{1}{\left(1+\frac{a^2}{r^2}+2\frac{a}{r}cos\,\theta\right)^{3/2}}\right]$$

$$\tilde{E}_\theta = \frac{Kqa\,sin\,\theta}{r^3}\left[\left(1-3\frac{a}{r}cos\,\theta\right)+\left(1+3\frac{a}{r}cos\,\theta\right)\right]=\frac{2Kqa\,sin\,\theta}{r^3}.$$

Then the vector electric field is $\tilde{\boldsymbol{E}} = \tilde{E}_r\hat{\boldsymbol{r}} + \tilde{E}_\theta\hat{\boldsymbol{\theta}} = \frac{2Kqa}{r^3}\left(3\,cos\,\theta\hat{\boldsymbol{r}} + sin\,\theta\hat{\boldsymbol{\theta}}\right)$.

As prompted by the activity, we calculate $-\nabla\tilde{V} = \frac{2Kqa}{r^3}\left(2\,cos\,\theta\hat{\boldsymbol{r}} + sin\,\theta\hat{\boldsymbol{\theta}}\right) \neq \tilde{\boldsymbol{E}}$. This means we should be careful not to assume that the approximate forms we found by taking the limits have all the properties of the exact functions. We see that they do not. We are assured that, for the exact expressions, $\boldsymbol{E} = -\nabla V$. However, the approximate forms need not satisfy this nice relationship – they are, after all, only approximations. With this caution in mind, would one expect that $\tilde{V} = -\int_\infty^r \tilde{\boldsymbol{E}} \bullet d\boldsymbol{s}$?

References

The first two textbooks in the following list are specifically referenced in the storyboard. The other entries represent a sampling of introductory physics textbooks that address ideas presented here. This list is certainly not exhaustive.

1. R. L. Finney and G. B. Thomas, *Calculus*, Second Edition, Addison-Wesley Publishing Company, 1994, ISBN 0-201-54977-8.

2. D. J. Griffiths, *Introduction to Electrodynamics*, Third Edition, Prentice-Hall, Inc., 1999, ISBN 0-13-805326-X.

3. J. Sanny and W. Moebs, *University Physics*, Wm. C. Brown Publishers, 1996, ISBN 0-697-05884-0.

4. R. A. Serway, *Physics for Scientists and Engineers*, Fourth Edition, Saunders College Publishing, 1997, Volume 1, ISBN 0-03-020043-1 and Volume 3, ISBN 0-03-020047-4.

5. R. L. Reese, *University Physics*, Brooks/Cole Publishing Company, 1999, ISBN 0-534-36964-2.