INTERDISCIPLINARY LIVELY APPLICATIONS PROJECT

Fixed Point Theorems

FIXED POINT THEOREMS IN ECONOMICS

Interdisciplinary Lively Applications Project

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Mathematics Classifications: Real Analysis

Disciplinary Classifications: Economics

Prerequisites:

1. The calculus of one and several variables. Specific concepts/techniques used include: differentiation; indefinite integration; continuity of real-valued functions of a single variable; optimization and constrained optimization in one and several variables.
2. Understanding of convergent and Cauchy sequences of real numbers.

Economic Concepts Examined: A macroeconomic model of capital accumulation.

Introduction

In 1979 David Blackwell (American, 1919-) of the University of California at Berkeley was awarded the TIMS/ORSA John von Neumann Prize. The citation begins:

"The John von Neumann Theory Prize for 1979 is awarded to David Blackwell for his outstanding work in developing the theory of Markovian decision processes, and, more generally, for his many contributions in probability theory, mathematical statistics, and game theory that have strengthened the methodology of operations research and management science.

"In the area of Markovian decision processes Blackwell, in a remarkable series of papers published between 1961 and 1966, put the theory of dynamic programming on a rigorous mathematical footing. He introduced new techniques of analysis and established conditions for the existence of optimal and stationary optimal policies. Particularly noteworthy are his studies of the effect of varying the discount rate and his introduction of the important concepts of positive and negative dynamic programs. Virtually all of the subsequent developments in this field are based on these fundamental papers."

Blackwell earned his Ph.D. in mathematics from the University of Illinois in 1941. He is the seventh African American to receive a Ph.D. in mathematics and, in 1965, became the first African American named to the National Academy of Sciences. He has published over 90 papers and books, and his 1954 book (with M.A. Girshick) *Theory of Games and Statistical Decisions* is classic. He has received many awards for his work, including the prestigious John von Neumann Prize. In this module, we will learn about some of his work in mathematical economics. The specific work that we will study appeared in his paper *Discounted dynamic programming* published in 1965 in the Annals of Mathematical Statistics; this is one of the papers in the "... remarkable series of papers published between 1961 and 1966".

Economists use known mathematical theory as an aid in addressing economic problems. The economic problems may come from any branch of economics – labor, international, or urban economics, etc., and the mathematics used may come from any branch of mathematics – geometry, differential equations, calculus, linear algebra, etc. The mathematical theory that we will use comes from the branch of mathematics known as *functional analysis* and the economic problem we address comes from *macroeconomics*.

This project is divided into three parts. In the first section you will learn the necessary functional analysis. You are asked to do several standard exercises so that you become more comfortable with this background material. In the second section you will learn about the economic problem and see models for it. Working with a specific example, you will be asked to verify certain aspects of the general model. In particular, you will solve the model analytically. In general, an analytic solution may be very difficult to solve for (we have chosen 'easy' functions to work with), and solutions are typically found numerically. In the third part of the project (the true 'project') you will be asked to design a method for numerically achieving this solution, and to compare this solution to the one derived analytically.

Section 1: The Mathematical Background

The primary goal of this section is to prove Banach's Contractive Mapping Principle (due to Stefan Banach, Polish, 1892-1945). This is an example of a *fixed point theorem*; fixed point theorems constitute a substantial branch of mathematics and have many applications in mathematics and in other areas, including mathematical economics. Most theorems that ensure the existence of solutions of differential equations can be reduced to fixed point theorems. Banach's theorem has its roots in the work of Augustin-Louis Cauchy (French, 1789-1867). Cauchy was working on solving a differential equation subject to a boundary condition:

$$\begin{cases} \dfrac{dy}{dx} = \phi(x, y) \\ y(x_0) = y_0. \end{cases}$$

‘Finding a solution’ means to construct a (necessarily continuous – can you explain why?) function $y(x)$ that passes through the point (x_0, y_0) with slope $\phi(x_0, y_0)$. Don't worry if you haven't taken a course on differential equations – you should have enough background from your calculus course.

Given the above system, how do we know a solution exists at all? If one exists, can we tell if it is the only one? Consider the following ‘easy’ example:

$$\begin{cases} \dfrac{dy}{dx} = y^{\frac{2}{3}} \\ y(0) = 0. \end{cases}$$

This, as you should check, has the two different solutions

$$y_1(x) = 0 \qquad \text{and} \qquad y_2(x) = \frac{1}{27}x^3.$$

Therefore, uniqueness does not always follow from existence.

In 1820, Cauchy proved the first uniqueness and existence theorems for a system of type

$$\begin{cases} \dfrac{dy}{dx} = \phi(x, y) \\ y(x_0) = y_0. \end{cases}$$

(The example above is indeed of this type, with $\phi(x, y) = y^{\frac{2}{3}}$ and $x_0 = 0 = y_0$.) However, he imposed severe restrictions on ϕ and the proof was unnecessarily complicated. There subsequently followed improvements on Cauchy's theorem and its proof, including an improvement of the proof due to (Charles) Emile Picard (French, 1856-1941). His proof employs what is now referred to as his ‘Picard's iterative method of successive approximations’.

As stated before, we aim to prove Banach's theorem; the proof we supply is a generalization of Picard's iterative method. We will then rephrase Cauchy's problem about differential equations into the language of the fixed point theorem. It might seem to you that this rephrasing makes a relatively simple problem from differential equations seem unnecessarily complicated and, perhaps, even obscure; the point is that by viewing any specific mathematical problem in a more general setting, we greatly increase the scope of applications.

The appropriate setting for the theorem is a *metric space*. A metric space is an example of what is more generally called a *topological space*.

A 'metric' is simply a way of measuring distances between points of the space. The space $\mathbb{R}^n$ is a metric space with metric

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2}$$

for $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{R}^n$. One can check that this metric satisfies each of the four conditions in the following definition.

Definition. *A* **metric space** (X, d) *is a set* X *together with a function* $d : X \times X \to \mathbb{R}$ *called a* **metric** *satisfying*

(a) $d(x,y) \geq 0$ *for all* $x, y \in X$ *(non-negativity)*

(b) $d(x,y) = 0$ *if and only if* $x = y$ *(non-degeneracy)*

(c) $d(x,y) = d(y,x)$ *for all* $x, y \in X$ *(symmetry)*

(d) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$ *(triangle inequality).*

Often (X, d) is referred to as X when the metric d is understood.

Some examples of metric spaces:

1. $\mathbb{R}$, with $d(x,y) = |x - y|$
2. Let ℓ^2 be the set of all infinite sequences of real numbers $\{x_n\}_{n=1}^{\infty}$ satisfying

$$\sum_{n=1}^{\infty} x_n^2 < \infty,$$

with

$$d(\{x_n\}, \{y_n\}) = \sqrt{\sum_{n=1}^{\infty} |x_n - y_n|^2}.$$

3. Fix a positive integer n and let X be the set of all ordered n-tuples of 0's and 1's. For x and y in X, define $d(x,y)$ to be the number of places in which x and y differ. For example, with $n=6$,

$$d(001011, 101001) = 2.$$

4. For $a < b$, let

$$C([a,b]) = \{f \mid f : [a,b] \to \mathbb{R} \text{ such that } f \text{ is continuous}\}$$

with

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\}(= \max\{|f(x) - g(x)| : x \in [a,b]\}).$$

5. For any set $X \subseteq \mathbb{R}$, let

$$B(X) = \{f \mid f : X \to \mathbb{R} \text{ such that } f \text{ is bounded}\}$$

with

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in X\}.$$

The last two spaces will be of most interest to us.

Exercise 1. *In $C([0,1])$, compute $d(f,g)$ and $d(h,g)$ for $f(x) = 1$, $g(x) = x$, and $h(x) = \sin(2x)$.*

You should note that metric spaces are not necessarily vector spaces. It is worth mentioning that we are always considering our vector spaces as over $\mathbb{R}$.

Exercise 2. *(We recommend assigning perhaps only parts of this problem.) Verify that the examples above are metric spaces. Four of these five metric spaces are in fact vector spaces; which ones? Explain.*

Many of the notions from $\mathbb{R}$ can be extended to the general metric space setting. Recall what it means for a sequence of real numbers to converge, and for a sequence to be Cauchy.

Definition. *Consider a sequence* $\{x_n\}$ *in a metric space* X. *It is said to* **converge** *to an element* $x \in X$ *if* $d(x_n, x) \to 0$ *as* $n \to \infty$. *It is said to be* **Cauchy** *if* $d(x_n, x_m)$ *can be made arbitrarily small by choosing* n *and* m *sufficiently large. If every Cauchy sequence of* X *converges in* X, *the space* X *is called a* **complete** *metric space.*

Exercise 3. *Prove that* $C([a,b])$ *with supremum norm (as in example 4 above) is a complete metric space.*

Exercise 4. *Prove that* $B(X)$ *with supremum norm (as in example 5 above) is a complete metric space.*

Before we proceed to the fixed point theorem, we discuss the "dimension" of a metric space. Certain spaces, like $\mathbb{R}^n$, are *finite dimensional.* A *basis* for $\mathbb{R}^n$ is given by the collection of vectors

$$
\begin{aligned}
e_1 &= (1, 0, 0, \cdots, 0) \\
e_2 &= (0, 1, 0, \cdots, 0) \\
e_3 &= (0, 0, 1, \cdots, 0) \\
&\cdot \\
&\cdot \\
&\cdot \\
e_n &= (0, 0, 0, \cdots, 1).
\end{aligned}
$$

That these vectors form a basis for $\mathbb{R}^n$ means

(a) they are linearly independent – that is, if $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ and $\alpha_1 e_1 + \alpha_2 e_2 + \cdots \alpha_n e_n = 0$, then, necessarily, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, and

(b) they span $\mathbb{R}^n$ – that is, every vector in $\mathbb{R}^n$ can be written as a linear combination of these basis vectors.

More generally, a vector space is said to be *n-dimensional* if the largest number of linearly independent elements is n. This, we assume, is familiar to you.

We've also looked at metric spaces, like ℓ^2 and $C([a,b])$, that are *infinite dimensional.* This means that for each positive integer n there exists a linearly independent subset containing n elements. Another way of saying this is that there is no finite subset whose linear combinations constitute the entire space.

Exercise 5. *Let V be the vector space of all $n \times n$ matrices. What's the dimension of V?*

Exercise 6. *Prove that $C([a,b])$ is infinite dimensional.*

Exercise 7. *Prove that ℓ^2 is infinite dimensional.*

We now turn our attention to our main theorem.

Definition. *Let $X = (X,d)$ be a metric space. A map $T : X \to X$ is a* **contraction** *if there exists an $M \in [0,1)$ such that $d(Tx,Ty) \leq M \cdot d(x,y)$ for all $x, y \in X$.*

We are now ready to prove

The Banach Contractive Mapping Theorem. *Let X be a complete metric space and T be a contraction $X \to X$. Then there exists a unique point $x \in X$ with $Tx = x$.*

Proof: Choose any $x_0 \in X$. Put

$$
\begin{aligned}
x_1 &= Tx_0 \\
x_2 &= Tx_1 (= T^2 x_0) \\
&\cdot \\
&\cdot \\
&\cdot \\
x_n &= T^n x_0.
\end{aligned}
$$

We will show that this defines a Cauchy sequence, $\{x_n\}$. Then, since X is complete, we know that this sequence converges to some element of X, which we'll call x. We will finish by showing that x is a fixed point of T and that it is the only fixed point of T.

To see that $\{x_n\}$ is Cauchy, we first note that

$$d(x_{n+1}, x_n) \leq M^n d(x_1, x_0).$$

Exercise 8. *Prove this inequality.*

Then, if $m > n$,

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\le (M^{m-1} + M^{m-2} + \cdots + M^n) d(x_1, x_0) \\ &\le (\frac{M^n}{1-M}) d(x_1, x_0). \end{aligned}$$

Since

$$\frac{M^n}{1-M} \to 0$$

as $n \to \infty$, we see that $\{x_n\}$ is Cauchy. Let $x = \lim_{n\to\infty} x_n$ and notice that

$$\begin{aligned} d(Tx, x) &\le d(Tx, Tx_n) + d(Tx_n, x) \\ &\le Md(x, x_n) + d(x_{n+1}, x). \end{aligned}$$

Since both

$$d(x, x_n) \to 0 \qquad \text{and} \qquad d(x_{n+1}, x) \to 0$$

as $n \to \infty$, we see that $d(Tx, x) = 0$. Thus, $Tx = x$. To complete the proof it remains to be shown that x is the *only* fixed point of T. Suppose that $Ty = y$ and that $x \ne y$. Then $d(x, y) > 0$ and thus

$$d(x, y) = d(Tx, Ty) \le Md(x, y) < d(x, y),$$

a contradiction. Therefore, it must be the case that $x = y$. This completes the proof. ■

We now return to our differential equation with boundary condition:

$$\begin{cases} \dfrac{dy}{dx} = \phi(x, y) \\ y(x_0) = y_0. \end{cases}$$

A solution to this system, if it exists, will be an element of the infinite dimensional metric space $C([a, b])$ for some closed interval $[a, b]$ containing x_0. This system is equivalent to the *integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} \phi(t, y(t)) dt.$$

Exercise 9. *Show that the above system and the above integral equation are equivalent, i.e., show that a function $y(x)$ satisfies one if and only if it satisfies the other.*

We will now use Banach's theorem to show that a unique solution to this integral equation exists. The theorem requires that we have a contraction mapping acting on some metric space. As stated above, a solution to this system, if it exists, will be an element of the infinite dimensional metric space $C([a, b])$ for some closed interval $[a, b]$ containing x_0. We thus use $C([a, b])$ for our metric space. The map we use is

$$T : C([a, b]) \to C([a, b])$$

where $Tf = g$ is given by

$$g(x) = y_0 + \int_{x_0}^{x} \phi(t, f(t))dt.$$

Any solution of the original differential system is a solution to the integral equation which in turn is a fixed point of the map T.

It is important to realize that this T may or may not be a contraction. Our theorem will only apply in the case that T is a contraction and so we search for conditions on T ensuring this. To this end, assume that ϕ *satisfies a Lipschitz condition* in the second variable. That is, we assume that there exists a positive number K such that

$$|\phi(x, y) - \phi(x, z)| \leq K|y - z|$$

for all $x \in [a, b]$ and all $y, z \in \mathbb{R}$. For $f_1, f_2 \in C([a, b])$ and $x \in [a, b]$, we have

$$\begin{aligned}
|(Tf_1)(x) - (Tf_2)(x)| &= \left| \int_{x_0}^{x} [\phi(t, f_1(t)) - \phi(t, f_2(t))]dt \right| \\
&\leq \int_{x_0}^{x} |\phi(t, f_1(t)) - \phi(t, f_2(t))|dt \\
&\leq \int_{x_0}^{x} K|f_1(t) - f_2(t)|dt \\
&\leq \int_{x_0}^{x} Kd(f_1, f_2)dt \\
&\leq K(b - a)d(f_1, f_2).
\end{aligned}$$

Since this holds for all $x \in [a, b]$,

$$d(Tf_1, Tf_2) \leq K(b-a)d(f_1, f_2)$$

for all $f_1, f_2 \in C([a, b])$. From this, we see that the map T is a contraction so long as $K(b-a) < 1$.

In some simple situations, Picard's method of successive approximation can actually be used to construct the unique solution to a differential equation with boundary condition. The next exercise illustrates this.

Exercise 10. *Consider the system:*

$$\begin{cases} \frac{dy}{dx} = x + y \\ y(0) = 0. \end{cases}$$

First, verify that the conditions of the Banach Contractive Mapping Theorem are satisfied and thus deduce that this system has a unique solution. Then, use Picard's method to show that $f(x) = e^x - x - 1$ is the solution to the system. To do this, choose some convenient f_0, define $f_{n+1} = Tf_n$, and show that the iterates $f_1, f_2, \ldots$ converge to $f(x) = e^x - x - 1$.

This exercise is designed so that the iterates $f_1, f_2, \ldots$ can be solved for quite easily, and so that it is not too hard to recognize the series one comes up with. In most cases, neither of these is apparent. Indeed, often,

i. the iterates can not be solved for with elementary functions, or are too hard to calculate, and

ii. even if we 'have' the iterates, to figure out what they converge to may be very difficult.

We reiterate that the power of this theorem, even though the proof given is a 'constructive' proof, is *theoretical*, i.e., it is primarily for *assuring* existence and uniqueness of the solution, not for determining what the solution actually is.

For this project, we are interested in different conditions under which the Banach Contractive Mapping Theorem is applicable. The conditions we will use are due to David Blackwell and first appeared as Theorem 5 of [2].

Theorem. *(Blackwell's sufficient conditions for a contraction) Let* $T : B([x_1, x_2]) \to B([x_1, x_2])$ *be an operator satisfying*

(a) (monotonicity) $f, g \in B([x_1, x_2])$ *and* $f(x) \leq g(x)$ *for all* $x \in [x_1, x_2]$*, implies* $(Tf)(x) \leq (Tg)(x)$ *for all* $x \in [x_1, x_2]$*;*

(b) (discounting) there exists some $\beta \in (0, 1)$ *such that*

$$[T(f + a)](x) \leq (Tf)(x) + \beta a$$

for all $f \in B([x_1, x_2])$, $a \geq 0$, *and* $x \in [x_1, x_2]$. *Here,* a *is the constant function taking on the value* a*; in particular,* $(f + a)(x) = f(x) + a$.

Then T is a contraction.

Exercise 11. *Prove this theorem. (Hint: use* $a = d(f, g)$*.)*

Section 2: The Economic Background

Economic theory can be broadly divided into two disciplines: microeconomics and macroeconomics, with all subfields falling into one or both categories. Microeconomics is primarily the study of the economic behavior of firms and individuals ("agents"), while macroeconomics is devoted to the study of the economy as a whole. In macroeconomics, economists attempt to interpret observations of economic data in the aggregate and to predict the consequences of alternative government economic policies. Many economists believe that the aggregate necessarily builds upon the actions of individuals and thus the line between micro- and macroeconomic theory becomes blurred.

Here, we present a macroeconomic model that relies on the economic choices made at the individual (or microeconomic) level. In particular, we are interested in how the behaviors of economic agents, given the constraints these agents face, affect economic data in the aggregate. The passage of time is a crucial component of many of the aggregates that economists study, for example, capital accumulation. In this module, we develop a simple macroeconomic model of capital accumulation and demonstrate how the use of functional analysis allows the analyst to solve seemingly intractable economic problems.

In our simple model, we assume that all economic agents in the economy are identical; given this assumption, we can consider a "representative agent" who will reflect the actions of all actors in the economy.[1] This agent (or consumer) will maximize her utility (or satisfaction) subject to a series of constraints. We assume that there is a single consumption good, c, that increases the satisfaction (or utility), U, of the consumer and that the consumer lives forever ("infinite time horizon"). A single good is produced in the economy through the use of a set technology, f, which is a function of the stock of physical capital in the economy, k.

A Simple Model of Optimal Growth

We consider time as varying discretely, i.e., t can only take on integer values. Consumption in time period t is designated by c_t and the stock of capital by k_t. We have a utility function, $U : [0, \infty) \to \mathbb{R}$, which takes a unit of consumption and translates it to a level of satisfaction or utility. We 'discount' utility in future periods by β to reflect the fact that consumption today is more satisfying to people than is consumption tomorrow. The production function, $f : [0, \infty) \to [0, \infty)$, takes a unit of capital and translates it into a good that can either be consumed today or invested in a capital stock for production in the future. We make the following assumptions about the utility and production functions, U and f:

$$\lim_{c \to 0^+} U'(c) = \infty, \quad U' > 0, \quad U'' < 0;$$

and

$$f(0) = 0, \quad \lim_{k \to \infty} f'(k) = 0, \quad \lim_{k \to 0^+} f'(k) = \infty, \quad f' > 0, \quad f'' < 0.$$

[1] This example is a standard beginning exercise in a first-year, graduate-level course in macroeconomic theory. Our treatment of this exercise is based in large part upon discussions of dynamic programming in Stokey and Lucas (1989) and in Sargent (1987).

Each of these assumptions makes sense, economically speaking. For example, $f(0) = 0$ means that zero capital results in no production; $f' > 0$ means that production increases as capital increases; $f'' < 0$ means that increasing capital increases production at a decreasing rate (this is called "diminishing returns to scale").

The representative consumer finds sequences $\{c_t^*\}_{t=0}^{\infty}$ and $\{k_{t+1}^*\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

subject to the following constraints

$$c_t + k_{t+1} = f(k_t), \quad \text{for each } t = 0, 1, 2, \ldots \tag{2}$$

$$c_t \geq 0, \quad \text{for each } t = 0, 1, 2, \ldots \tag{3}$$

where $k_0 > 0$ is given and $0 < \beta < 1$. That is, she is trying to find sequences $\{c_t^*\}_{t=0}^{\infty}$ and $\{k_{t+1}^*\}_{t=0}^{\infty}$ such that

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \leq \sum_{t=0}^{\infty} \beta^t U(c_t^*)$$

for all other sequences $\{c_t\}_{t=0}^{\infty}$, and $\{k_{t+1}\}_{t=0}^{\infty}$ satisfying

$$c_t + k_{t+1} = f(k_t),$$

and

$$c_t \geq 0,$$

with a fixed $k_0 > 0$ and specified $0 < \beta < 1$.

Substituting the budget constraint (equation (2)) into equation (1), we can restate the problem as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \tag{4}$$

subject to

$$0 \leq k_{t+1} \leq f(k_t) \tag{5}$$

where k_0 is given.

Let us step back for a moment to consider a simple case with a finite time horizon, T. In this case, equations (4) and (5) become a standard concave programming problem. In other words, once we consider specific functions U and f, we can use the tools of constrained optimization developed in multivariable calculus (such as Lagrange multipliers) in order to find the maximizing sequences of consumption, $\{c_t\}_{t=0}^T$, and capital, $\{k_{t+1}\}_{t=0}^T$.

For the next exercise, we consider specific utility and production functions. These are: $U(c) = \ln(c)$ and $f(k) = k^\alpha$, for some $0 < \alpha < 1$.

Exercise 12. *Confirm that the evolution of capital, given by*

$$k_{t+1} = \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} k_t^\alpha, \quad t = 0, 1, 2, \ldots, T, \tag{6}$$

with a given k_0, satisfies the problem defined by equations (4) and (5), in the finite horizon case. Hint: Confirm that this capital accumulation function satisfies the first order condition

$$U'[f(k_t) - k_{t+1}] = \beta f'(k_{t+1}) U'[f(k_{t+1}) - k_{t+2}], \quad t = 1, 2, 3, \ldots, T,$$

and boundary condition

$$k_{T+1} = 0, \quad k_0 > 0 \text{ given.}$$

Can you explain why this is sufficient?

Taking the limit of equation (6) as $T \to \infty$ gives (make sure you understand why!)

$$k_{t+1} = \alpha\beta k_t^\alpha. \tag{7}$$

One might be led to conclude from this that the solution to the infinite time horizon problem would always be of the form

$$k_{t+1} = g(k_t), \quad t = 0, 1, 2, \ldots \tag{8}$$

where $g : [0, \infty) \to [0, \infty)$ is the savings function for capital. This solution, however, is specific to the functions $U(c) = \ln(c)$ and $f(k) = k^\alpha$, $0 < \alpha < 1$. It is not readily apparent how to generalize this solution for general U and f.

Let us limit the problem the consumer faces to choosing today's consumption and tomorrow's capital, c_0 and k_1, respectively. Then define a value function, $v : [0, \infty) \to \mathbb{R}$, by taking $v(k_0)$ to be the value of the maximized objective function (equation (4)) for each possible value of $k_0 > 0$. Then $v(k_1)$ is the value of the utility the consumer receives in period 1, with capital stock k_1. We can now restate the problem of the consumer in the following way.

$$\max_{c_0, k_1} \{U(c_0) + \beta v(k_1)\} \tag{9}$$

subject to

$$c_0 + k_1 = f(k_0) \tag{10}$$

$$c_0 \geq 0, \quad k_1 \geq 0 \tag{11}$$

where $k_0 > 0$ is given. If $v(k)$ were known, we could then define a function, $g : [0, \infty) \to [0, \infty)$, such that for each $k_0 > 0$, $k_1 = g(k_0)$ and $c_0 = f(k_0) - g(k_0)$, where c_0 and k_1 are the values that maximize equation (9). This function, g, then would describe the capital accumulation in this economy for any initial capital stock, k_0. Since we define v to be the maximized objective function (equation (4)), it follows that

$$v(k_0) = \max_{c_0, k_1} \{U(c_0) + \beta v(k_1)\} \tag{12}$$

subject to

$$c_0 + k_1 = f(k_0) \tag{13}$$

or, substituting (13) in to (12),

$$v(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \{U(f(k_0) - k_1) + \beta v(k_1)\} \qquad (14).$$

We refer to this type of problem formulation as "recursive", and we no longer require the use of subscripts. Thus, it is convenient to restate the recursive problem as the functional equation

$$v(k) = \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta v(k')\}. \tag{15}$$

Note that here k' represents capital in the next period, not a derivative! This use of notation is standard in the economics literature. Although we have given some indication, we haven't proved that the solution(s) to the problem posed in terms of the infinitely-lived consumer (equations (1), (2), and (3)) are the same as the solution(s) to the functional equation (15). For a formal proof, see [6] pages 67-77. We take this proof as given.

It has been shown by Benveniste and Scheinkman [3] that under a broad set of conditions, the value function v is differentiable (in this case with $v'(k) = f'(k)U'(f(k) - k')$ where $k' = g(k)$). One set of sufficient conditions is that the utility function, U, is concave and bounded and also that the constraint equations are each convex. Since these conditions are met in this case, we are assured the existence of the derivative of the value function v. Then the first order conditions are as follows:

$$U'(f(k) - g(k)) = \beta v'(g(k)) \tag{16}$$

$$v'(k) = f'(k)U'(f(k) - g(k)), \tag{17}$$

where $g(k)$ is the maximizing value of k'. The economic interpretations of these equations are as follows. Equation (16) states that the additional utility the consumer receives from consuming an additional unit of consumption today is equal to the discounted utility she would receive if she invested the unit in capital and then consumed it in the next period. Equation (17) indicates that the value of an additional unit of current capital is equal to the additional utility gained by allocating the current capital to production and then consuming the resulting product.

Exercise 13. *As before, assume* $U(c) = \ln c$ *and* $f(k) = k^{\alpha}$, $0 < \alpha < 1$. *Use equations (4) and (7) to solve for* $v(k)$. *Confirm that the function* v *satisfies (15), i.e., that it satisfies (16) and (17).*

In this simple example, we can solve for g analytically (as we did in (7)), but for most economic problems it is impossible to generate an analytic solution, and the problem must be solved numerically using an algorithm based upon (15). Once v is approximated numerically, g is also approximated.

Exercise 14. *Show that* g *is a nondecreasing function of* k. *(Hints: Since* U *and* f *are strictly concave,* v *is strictly concave.*[2] *Also, recall that* v *is differentiable with* $v'(k) = f'(k)U'(f(k) - k')$ *where* $k' = g(k)$.*)*

There is a maximal level of capital stock that can be sustained in this economy in the steady state, where the steady state occurs when $k = k'$, and $c = c'$, i.e., none of the variables evolve or change. This maximal level is achieved when $c_t = 0$ and $k_{t+1} = f(k_t)$ for all t. Call this maximal level, $\hat{k}$. Given the properties of f, the equation $\hat{k} = f(\hat{k})$ has a unique positive solution, and $k_{t+1} = f(k_t)$ converges to $\hat{k}$ as $t \to \infty$. (To check, plot $f(k_t)$ against a 45 degree line.) Let $k_0 \in [0, \hat{k}]$. Then k_t remains in the bounded interval $[0, \hat{k}]$ for all t.

Exercise 15. *Given that the optimal policy function,* g*, is nondecreasing in* k *(which we know from exercise 14), and is bounded in the interval* $[0, \hat{k}]$ *(which we know from the preceeding paragraph), show that the sequence* $\{k_t\}_{t=0}^{\infty}$ *is a monotone and bounded sequence.*

Since monotone, bounded sequences converge, it follows that k_t will converge to a limit point, k^*. Further, $g(k^*) = k^*$, i.e., k^* is a fixed point of g.

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[2] *A real-valued function* ϕ *is said to be a strictly concave function of* $x \in I$ *where* I *is some interval in* $\mathbb{R}$, *if* $\phi[\theta x + (1-\theta)y] > \theta\phi(x) + (1-\theta)\phi(y)$ *for every* $x, y \in I$, $x \neq y$, *and every* $0 < \theta < 1$.

Thus, by considering the functional equation (15) and the difference equation (8) that characterizes the optimal policy function, we are able to analyze a deterministic, infinite-horizon optimization problem. (For example, this method of analysis would allow us to consider rates of convergence to the steady state or fixed points of g.)

Next we must establish the existence and uniqueness of the value function, v. To do so, we will define an operator T which satisfies Blackwell's condition, and then invoke the Contraction Mapping Theorem.

In this example, we guessed the optimal policy function, g. Since knowing g is equivalent to knowing v, this is the same as guessing v. With practice and under certain types of models, guessing may be a feasible way to start, but under most circumstances it is impractical. Another approach is the method of successive approximations. Take an initial guess at a specific function that would satisfy (15), v_0. Define v_1 to be

$$v_1 = \max_{0 \le k' \le f(k)} \{U(f(k) - k') + \beta v_0(k')\}. \tag{18}$$

If $v_0(k) = v_1(k)$ for all $k > 0$, you have found your solution. Otherwise define

$$v_2 = \max_{0 \le k' \le f(k)} \{U(f(k) - k') + \beta v_1(k')\}. \tag{19}$$

Continue to define functions recursively:

$$v_{n+1}(k) = \max_{0 \le k' \le f(k)} \{U(f(k) - k') + \beta v_n(k')\}, n = 0, 1, 2, \ldots. \tag{20}$$

We can show that as n increases, v_n gets closer to the true value function, v. Suppose we give an initial guess for the policy function g_0, where $0 \le g_0 \le f(k)$, for all $k > 0$. Then (18) becomes a problem in which the consumer can choose the maximizing value of k' in the first period, but is then constrained to follow $k' = g_0(k)$ for all subsequent periods. Since g_0 is a feasible choice in the first period, the consumer will receive at least as high a utility level as she would if she had chosen g_0. Therefore, $v_1(k) \ge v_0(k)$. By induction, we can show that $v_{n+1}(k) \ge v_n(k)$ for all k and $n = 0, 1, 2, \ldots$.

For any value function, $v : [0,\infty) \to \mathbb{R}$, define a new function $Tv : [0,\infty) \to \mathbb{R}$, to be

$$(Tv)(k) = \max_{0 \le k' \le f(k)} \{U(f(k) - k') + \beta v(k')\}. \tag{21}$$

This rule assigns a map T from the set of continuous functions into itself, i.e., $T : C([0,\infty)) \to C([0,\infty))$. Then, the fixed point of T, i.e., some function $v \in C([0,\infty))$ such that $v = Tv$, will be the solution to equation (15). In this application, Blackwell's sufficient conditions for a contraction mapping can be easily verified. Thus, T is a contraction mapping and our theorems assure the existence and uniqueness of the solution. Further, the method of successive approximations provides a means for constructing approximations to this solution.

Exercise 16. *Verify that this T satisfies Blackwell's conditions.*

Section 3: The Project

Consider the problem outlined in the ILAP module. Assume $U(c) = \ln c$ and $f(k) = k^\alpha$. Let $\alpha = .3$ and $\beta = .95$.

1. Solve for the steady state level of capital (the fixed point of g) using (7).

2. Set up a grid for the capital stock, making sure that the steady state is in that grid. (By grid, we mean that you will create 2 nested do loops: the outer one for k and the inner one for k'.) Once you're in the do loops, figure out which pairs of k and k' are infeasible. (By infeasible, we mean that $k' \ge k^\alpha$.) If you are in an infeasible pair, go to the end of the do loop. Then, for each feasible pair (k, k') solve the objective function and store it. Since you will have substituted out for c, k' will be the only choice variable. So, this is an easy case in which you just plug the (k, k') pair into the objective function. Store these "return" functions, say in $rr(i, j)$ where $i = k$ and $j = k'$.

Give an initial guess for v, say $vin(i) = 0$. So, vin will be a vector with the same number of elements as the vector k. Create do loops over k, k'. Then, for each feasible

(k, k') pair solve for $vnew(i) = rr(i, j) + \beta * vin(i)$ and choose the k, k' pair that maximizes $vnew(i)$ for each i.

Then create another do loop over k. Compare $vnew(i)$ to $vin(i)$. If they're the same (you can decide how close you want them to be; start with a large difference, say $d = .001$, until you have the program up and running) for all i, then you have your solution. If not, then let $vin(i) = vnew(i)$ and go back to the prior loop. In other words, for the first k, iterate over the k' to find the k' that gives the maximum value of $vnew(k) = rr(k, k') + \beta * vin(k')$. Then go to the next k and do the same, and so on through the grid. Once you've done it for the entire grid, you will have a vector, $vnew(k)$. Then compare $vnew(k)$ to $vin(k)$ for each k. If they are the same, you have the solution to the value function. If not, substitute $vnew$ in for vin and then repeat until $|vnew - vin| < d$.

Our advice would be to set a fat grid to start with, until you know the program is running. Then narrow it to get a more precise solution. The solution to this problem is twofold: the value function v and the policy function g. In this case, we are able to generate an analytic solution to the policy function, which is $k' = \alpha\beta k^{\alpha}$. Steady state capital is $(\alpha\beta)^{\frac{1}{1-\alpha}}$.

3. When you are done solving the problem, plot $g(k)$ and the 45 degree line. Then, you should be able to see where the steady state is. As a final check, plot the difference between the policy function that you estimated and the analytic solution.

4. Write a report of your project.

Instructions for your Written Report

Effective writing is not easy, and we hope that you take the opportunity to develop your skills in this area seriously. The following, meant as a general guide,[3] may prove useful.

In advance of writing your paper, you must decide on your *audience*. For example, you could choose to write at a level appropriate for your teacher, or you may choose to write to your classmates. Make this choice, and stick to it!

[3] This guide is a modified version of the one by Kent M. Miller found in [1].

You should include:

I. Title Page. Should display, attractively, the title of the project, the name(s) of author(s), the date, the name and number of the course, and the name of the instructor.

II. Introduction. The purpose of the introduction is to briefly and clearly describe the problem and to summarize the solution(s) and solution method(s). The introduction should be a "stand alone" document, no more than two pages in length. Attempt to engage your reader. Your instructor should evaluate your summary in terms of style, substance, organization, and correctness.

III. Main Body of the Report. Include one subsection for each major requirement of your project. The first paragraph of each subsection should be a concise statement of the problem. Next, give the facts having influence on the problem or its solution. Exercise care to exclude unnecessary facts that may confuse the issue. Next, state any assumptions necessary to solve the problem. Give definitions of variables and symbols you will use. Be organized, systematic, unambiguous. Describe your methodology and your calculations (hand and/or computer). The main body of the paper may include essential graphs, plots, computer output, data sets, etc., but don't overdo it. Make sure that these are labelled, and that they are referred to in the text.

IV. Summary and Conclusions. Conclusions should follow logically from your work explained in the main body of your paper. Do not introduce any new material here, and try to answer the problem directly and efficiently. Summarize what you have accomplished. You may, also, comment on any surprising aspects of the project.

V. Appendices. Include here any supporting graphs, plots, computer output, data sets, etc., that do not appear already in the paper. Make sure that these are labelled, and that they are referred to in the text.

VI. References and Acknowledgements. Give full bibliographic information for any books, papers, web pages, personal conversations used.

To The Instructor

This module is written for use either in a first course on real analysis, a second course in real analysis, functional analysis, a course on computational or applied mathematics. While the only mathematical background necessary is the calculus of one and several variables, and a familiarity with matrices, the material found in this module is difficult and therefore requires a certain 'mathematical sophistication' on the part of the reader. No background in economics is required, or assumed.

Our goal in writing this module is to introduce to math students to *one* methodology used by economists to prove the existence and uniqueness of an equilibrium. There are other methods; however, we have focused on one very particular, and basic, example in this paper. We end the exercise by asking the student to prove the existence and uniqueness of the equilibrium, and then also to use numerical methods to approximate it. The economist would use the model to conduct policy experiments; we do not discuss this applied aspect at all.

The first section introduces the mathematics. We lecture on this material - it takes 1-2 hours of class time. Alternatively, the text may be given as a reading assignment. Exercises are included. If they are not done, the facts that they assert must be accepted.

The second section introduces the economic theory. We give this as a reading assignment, though it too may be presented in class. Again, exercises are included.

The third section contains a description of, and instructions for, the project. A small amount of programming is required.

Following this, you will find a section containing solutions to all of the exercises, and to the project. We have also included a sample *Mathematica* program. Writing this program is part of the project.

We give the students, working in groups of two or three, two to three weeks to do the entire project.

A final word about the voice we adopt: we are writing to the *student*. When we write 'you', we mean the student reader.

Solutions to exercises and project

Exercise 1. *$d(f,g)$ is equal to the maximum difference between the two functions. Since $|f-g|$ varies between 1 and 0, $d(f,g)=1$. For the second difference, one must work a little bit harder: using calculus, one finds that the maximum occurs at $\frac{\pi}{6}$, and thus $d(g,h)=|\sin(2x)-x|=|\sin(\frac{\pi}{3})-\frac{\pi}{6}|$.*

Exercise 2. *The third example is not a vector space – you cannot multiply elements of X by scalars (and stay in X).*

Exercises 3 and 4. *You may want your students to do only parts of exercises 3 and 4. For example, you may choose that they prove completeness.*

Exercise 5. *n^2.*

Exercise 6. *Let $p_n(x)=x^n$ for $n=0,1,\ldots$. Each of these polynomials is an element of $C([a,b])$ and, by well-known properties of polynomials, the set $\{p_1,p_2,\ldots,p_n\}$ is linearly independent.*

Exercise 7. *Consider the sequence*

$$e_k=(0,0,\ldots,1,0,\ldots)$$

where 1 is placed in the kth place and there are zeros in all other places. It suffices to show that, for each positive integer n, the vectors $e_1,e_2,\ldots,e_n$ are linearly independent. This set is linearly independent since

$$\lambda_1e_1+\lambda_2e_2+\ldots+\lambda_ne_n=(\lambda_1,\lambda_2,\ldots,\lambda_n)$$

and so

$$0=(\lambda_1,\lambda_2,\ldots,\lambda_n)$$

is equivalent to

$$0=\lambda_1=\lambda_2=\ldots=\lambda_n.$$

Exercise 8. *We proceed by induction. For $n = 1$ we have:*

$$d(x_2, x_1) = d(Tx_1, Tx_0) \leq Md(x_1, x_0).$$

Assume that the statement holds for $n > 1$, i.e., that

$$d(x_n, x_{n-1}) \leq M^{n-1} d(x_1, x_0).$$

Then

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq M \cdot M^{n-1} d(x_1, x_0).$$

Exercise 9. *If we assume*

$$\begin{cases} \dfrac{dy}{dx} = \phi(x, y) \\ y(x_0) = y_0, \end{cases}$$

then integrating the first equation from x_0 to x leads to

$$\int_{x_0}^{x} y'(t)dt = \int_{x_0}^{x} \phi(t, y(t))dt.$$

Since y is an antiderivative of y', the Fundamental Theorem of Calculus tells us that

$$y(x) = y_0 + \int_{x_0}^{x} \phi(t, y(t))dt.$$

Conversely, differentiating both sides of

$$y(x) = y_0 + \int_{x_0}^{x} \phi(t, y(t))dt$$

yields

$$\frac{dy}{dx} = \phi(x, y);$$

evaluating

$$y(x) = y_0 + \int_{x_0}^{x} \phi(t, y(t))dt$$

at $x = x_0$ yields $y(x_0) = y_0$.

Exercise 10. *This system is equivalent to the integral equation*

$$f(x) = \int_0^x (t + f(t))dt,$$

so we put

$$(Tf)(x) = y_0 + \int_{x_0}^x \phi(t, f(t))dt$$

where $y_0 = 0, x_0 = 0$ *and* $\phi(t, f(t)) = t + f(t)$*. Is* ϕ *Lipschitz? Well,*

$$|\phi(x, y) - \phi(x, z)| = |(x + y) - (x + z)| = |y - z|$$

which is $\leq K|y - z|$ *if we put* $K = 1$*. Since we need* $K(b - a) < 1$ *in order to apply our theorem, we just need to restrict attention to an interval* $[a, b]$*, containing* $x_0 = 0$*, of length less than one. Then our theorem implies the existence and uniqueness of a solution; what is this solution? Let's choose, quite naively,* $f_0(x) = 0$*. Then*

$$f_1(x) = (Tf_0)(x) = \int_0^x (t + f_0(t))dt = \frac{1}{2}x^2$$

$$f_2(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$\vdots$$

$$f_n(x) = \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{(n+1)!}x^{n+1}.$$

From this we see that $f_n(x) \to f(x) = e^x - x - 1$ *and, sure enough,* $e^x - x - 1$ *solves the original system.*

Exercise 11. *First, we note that* $f \leq g + d(f, g)$ *and so monotonicity implies*

$$(Tf)(x) \leq (T(g + d(f, g)))(x)$$

for all $x \in [x_1, x_2]$. Discounting then implies that this is less than or equal to

$$(Tg)(x) + \beta d(f, g)$$

and so

$$(Tf)(x) - (Tg)(x) \leq \beta d(f, g).$$

Reversing the roles of f and g yields

$$(Tg)(x) - (Tf)(x) \leq \beta d(f, g)$$

and we thus have

$$|(Tf)(x) - (Tg)(x)| \leq \beta d(f, g)$$

for all $x \in [x_1, x_2]$, i.e., that

$$d(Tf, Tg) \leq \beta d(f, g).$$

Since $0 < \beta < 1$, we are done.

Exercise 12. *Since $U'(c) = \frac{1}{c}$ and $f'(k) = \alpha k^{\alpha-1}$, we can restate the first order condition with the appropriate functional forms:*

$$\frac{1}{k_t^\alpha - k_{t+1}} = \beta \frac{1}{k_{t+1}^\alpha - k_{t+2}} \alpha k_{t+1}^{\alpha-1}.$$

Substituting in the expression for k_{t+1} given in (6) we get,

$$\frac{1}{(1 - \alpha\beta\frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}})k_t^\alpha} = \frac{\alpha\beta}{(1 - \alpha\beta\frac{1-(\alpha\beta)^{T-(t+1)}}{1-(\alpha\beta)^{T-(t+1)+1}})k_{t+1}}.$$

Cross multiplying yields

$$k_{t+1} = \alpha\beta \left(\frac{1 - \alpha\beta\frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}}{1 - \alpha\beta\frac{1-(\alpha\beta)^{T-(t+1)}}{1-(\alpha\beta)^{T-(t+1)+1}}} \right) k_t^\alpha,$$

which reduces to

$$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha,$$

as desired. Setting $t = T$ makes it clear that the desired boundary condition, $k_{T+1} = 0$, is met.

This first order condition is sufficient because the maximum must occur where the derivative is zero; differentiating the sum in (4) with respect to k_{t+1} and setting the derivative equal to zero gives this first order condition.

Exercise 13. *Substituting (7) and the functional forms for U and f into (4) yields*

$$\sum_{t=0}^{\infty} \beta^t \ln(k_t^\alpha - \alpha\beta k_t^\alpha)$$

or

$$\sum_{t=0}^{\infty} \beta^t (\ln(1-\alpha\beta) + \alpha \ln(k_t)).$$

The first few terms of the above sum are as follows

$$\ln(1-\alpha\beta) + \alpha\ln(k_0) + \beta\ln(1-\alpha\beta) + \alpha\beta\ln(\alpha\beta k_0^\alpha) + \ldots$$

Using (7), and basic facts about geometric series, we can write this in closed form as

$$\frac{1}{1-\beta}\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \frac{\alpha}{1-\alpha\beta}\ln(k_0).$$

Thus,

$$v(k_0) = \frac{1}{1-\beta}\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \frac{\alpha}{1-\alpha\beta}\ln(k_0).$$

Next, we confirm that v satisfies (15), i.e., satisfies (16) and (17). Using the above calculated v, (16) holds if and only if

$$\frac{1}{(1-\alpha\beta)k^\alpha} = \frac{\alpha\beta}{1-\alpha\beta} * \frac{1}{\alpha\beta k^\alpha},$$

which is clear. And (17) holds if and only if

$$\frac{\alpha}{1-\alpha\beta} * \frac{1}{k} = \frac{\alpha k^{\alpha-1}}{(1-\alpha\beta)k^\alpha},$$

which is also clear.

Exercise 14. *Since U and f are strictly concave, it follows that v is strictly concave. Since f, f', and g are continuous, v is continuous. Consider k_j, j=1,2 and $U'(f(k_j) - g(k_j)) = \beta v'(g(k_j))$. Assume that $k_1 \geq k_2$ and $g(k_1) < g(k_2)$. The continuity of v' and the concavity of v ensure that $v'(g(k_j))$ is well-defined for all $g(k_j)$. Then,*

$$g(k_1) < g(k_2) \Rightarrow v'(g(k_1)) > v'(g(k_2)).$$

Therefore,

$$U'(f(k_1) - g(k_1)) > U'(f(k_2) - g(k_2)).$$

Since U is strictly concave,

$$U'(f(k_1) - g(k_1)) > U'(f(k_2) - g(k_2)) \Leftrightarrow f(k_1) - g(k_1) < f(k_2) - g(k_2)$$

or

$$0 < g(k_2) - g(k_1) < f(k_2) - f(k_1) \leq 0,$$

which contradicts the assumption that for $k_1 \geq k_2$, $g(k_1) < g(k_2)$.

This solution is described in Sargent [5] and is attributed to Sargent's then graduate student Rodolfo Manuelli.

Exercise 15. *Given g is a non-decreasing function in k and is bounded in the interval $[0, \hat{k}]$, show that the sequence $\{k_t\}_{t=0}^{\infty}$ is a monotone and bounded sequence.*

Let $k_0 \in [0, \hat{k}]$. Then for any $t > 0$, since g is bounded in the interval $[0, \hat{k}]$, $k_t \in [0, \hat{k}]$. Therefore, $\{k_t\}_{t=0}^{\infty}$ is a bounded sequence. Suppose that $k_1 > k_0$. Then, since $g(k)$ is non-decreasing, $k_2 = g(k_1) \geq g(k_0) = k_1$. Also, $k_3 = g(k_2) \geq g(k_1) = k_2$, and so forth. Next suppose that $k_1 < k_0$. Then, $k_2 = g(k_1) \leq g(k_0) = k_1$, and $k_3 = g(k_2) \leq g(k_1) = k_2$, and so forth. Thus, it follows that $\{k_t\}_{t=0}^{\infty}$ is a monotone and bounded sequence.

Exercise 16. *Monotonicity should be clear from the definition of T. The following verifies discounting:*

$$\begin{aligned} T(v+a)(k) &= \max_{0\le k'\le f(k)} \{U(f(k)-k') + \beta(v(k')+a)\} \\ &= \max_{0\le k'\le f(k)} \{U(f(k)-k') + \beta v(k') + \beta a\} \\ &= \max_{0\le k'\le f(k)} \{U(f(k)-k') + \beta v(k')\} + \beta a \\ &= Tv(k) + \beta a. \end{aligned}$$

Outline of a solution to the programming project.

1. $k = (\alpha * \beta)^{\frac{1}{1-\alpha}} = .1664$
2. *Here, the student should write a program where one can input six pieces of information.*

- $U(c) = \ln(c)$
- $f(k) = k^\alpha$
- α
- β
- $[a,b]$: an interval of input for $V([a,b])$
- the gridsize or the number of points (to be evenly distributed) in the interval $[a,b]$.

   The program must be designed to output the numerical values of $g(k)$, for

$$k = a,\ a + \text{gridsize},\ a + 2 * \text{gridsize},\ \ldots,\ b.$$

   Below is an example of a *Mathematica* program that one of our students wrote to solve this problem. The student chose an interval equal to $[.16, .17]$ and the number of points in the interval, max$K$, to be equal to 75. He set the gridsize to 1/7500. A finer gridsize (hence a greater number of points) yields a more accurate solution to the problem. Unfortunately, there is a tradeoff between accuracy and speed. As the number of grid points increases, the running time of the program increases.

3. Once the student has plotted $g(k)$ and the 45 degree line, the steady state is found at the intersection of the two graphs. Plotting the numerical and analytical solutions on the same set of axes gives the student a sense for how well the program solves the problem. If the student runs the program for different gridsizes, the increase in accuracy associated with finer grids becomes quickly evident.

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